

DETERMINATION OF THE MASS PROPERTIES  
OF A HUMAN IN ANY POSITION

by

ROBERT DANIEL WATKINS, B.S.

3894AN.

THESIS

Presented to the Faculty of the Graduate School of  
The University of Texas at Austin  
in Partial Fulfillment  
of the Requirements  
for the Degree of

MASTER OF SCIENCE IN ENGINEERING

**DISTRIBUTION STATEMENT A**

Approved for Public Release  
Distribution Unlimited

THE UNIVERSITY OF TEXAS AT AUSTIN

January 1975

**BEST AVAILABLE COPY**

20050531 142

## ACKNOWLEDGMENTS

For their encouragement and generous support, I thank my parents, Maj. Gen. James H. and Kathleen Watkins. Warm appreciation goes to my advisor, Dr. Wallace T. Fowler, for his suggestion that I enter graduate school and for his kind, thorough, and freely offered guidance. I also thank Dr. Byron D. Tapley for his ready and capable assistance on this thesis. In addition, I would like to acknowledge the support of the Engineering Foundation of The University of The University of Texas for their support of this project.

Robert Daniel Watkins

The University of Texas at Austin  
November 6, 1974

## ABSTRACT

In the past, human body models have been developed by assuming simple geometric shapes for the components of the human body in order to predict the mass properties, i.e. mass, center of mass, and inertia tensor. In this study a new personalized method of predicting component mass properties is developed, based on experimental data for whole-body mass properties for three body positions where only one component is moved. Also needed, however, are estimates for the mass and sum of the moments of inertia of each component.

Additionally, the contribution of outer garments or a suit to the mass properties can be included in the prediction if it is assumed that the components of the outer covering are rigid bodies that move negligibly relative to corresponding body components.

# TABLE OF CONTENTS

	Page
Acknowledgments . . . . .	iii
Abstract. . . . .	iv
List of Figures . . . . .	vii
Chapter 1 Introduction . . . . .	1
Purpose . . . . .	1
Background . . . . .	2
Study Structure . . . . .	3
Chapter 2 Determination of Whole-Body Mass Properties from Component Mass Properties with Respect to Component-Fixed Axes . . .	5
Notation . . . . .	5
Analysis . . . . .	9
Chapter 3 The Hanavan Model . . . . .	13
Chapter 4 Determination of Component Mass Properties about Component- Fixed Axes . . . . .	16
Overview . . . . .	16
Component Centers of Mass. . . . .	17
Component Inertia Tensors. . . . .	25
Experimental Determination of Whole-Body Mass Properties . . . . .	36
Consideration of Outer Garments or Suit . . . . .	37
Chapter 5 Explanation of the Infinity of Solutions for Component Inertia Tensors . . . . .	39
Chapter 6 Suggested Methods for Estimation of Component Masses . . .	43
Regression Equations . . . . .	43
Water Immersion. . . . .	45
Inclusion of Outer Garments or Suit in Component Masses . . . . .	46

	Page
Chapter 7 Suggested Methods for Estimation of the Sums of Component Moments of Inertia . . . . .	47
Hanavan Model. . . . .	47
Casts of Subject's Components . . . . .	48
Inclusion of Outer Garments or Suit in $s_j$ . . . . .	48
Conclusion . . . . .	49
Chapter 8 Limitations. . . . .	50
Invalidity of Assumptions . . . . .	50
Possible Errors in Whole-Body Experimental Data . . . . .	52
Accumulated Errors in Central Component . . . . .	52
Joint Constraints . . . . .	53
Suggested Sensitivity Analyses . . . . .	53
Chapter 9 Conclusion . . . . .	55
Appendix A Computer PROGRAM WHOLE. . . . .	58
Appendix B Computer PROGRAM PART . . . . .	62
Bibliography. . . . .	71
Vita	

## LIST OF FIGURES

Figure		Page
1	Vectors and Coordinate Systems . . . . .	8
2	Rotation Angles. . . . .	11
3	The Hanavan Model. . . . .	14
4	Flow Chart of the Determination of Component Mass Properties . .	57

## CHAPTER 1

### Introduction

#### Purpose

The motion of a dynamic system depends on the mass properties of the system, i.e. the mass, center of mass, and inertia tensor. If a human being is a part of the system, then the system properties depend on his or her contribution. The human's contribution to mass properties depend on body position, and hence become a function of time if the human is in motion.

For many dynamic systems the human's contribution may be negligible, as for a large airplane, where the pilot's mass is small compared with the mass of the system. However, there are systems where this is not the case, for example an astronaut with a pressure suit and backpack. For some dynamic systems, the human may constitute the entire system, e.g. a swimmer, a diver, a gymnast. Where the human's contribution to the system is significant, a prediction of the system dynamics depends upon knowledge of the time history of the person's mass properties. While a human's mass is directly measurable (using scales), this is not the case for the other mass properties.

The purpose of this study is to develop a method for predicting the center of mass and the inertia tensor for any human being in any given position, with or without outer garments or suits. Thus, in this study, methods are presented for obtaining the time history of the mass properties of any particular human being directly from the time history of the body position.

The means presented for predicting body center of mass and inertia tensor shall depend upon the following assumptions:

(1) The human body is adequately modeled by a system of rigid bodies connected by joints.

(2) There exist valid method for estimating the mass of each body component. A review of possible methods is given in Chapter 6.

(3) There exist valid method for estimating the sum of the three moments of inertia for each component about component-fixed axes. Possible methods are discussed in Chapter 7.

Assumption (1) above is equivalent to assuming that body components do not deform appreciably for different body positions. This assumption is questionable for the torso, whose curvable, twistable backbone can make seemingly significant changes in torso shape. For a suited, backpacked astronaut this assumption seems more reasonable because of the restricted mobility imposed by currently used equipment.

Assumption (2) arises as a consequence of the fact that there appears to be no nondestructive means for experimentally determining the mass of the individual body components. There seems no way to directly weigh the individual body components of a living human because of the unknown forces at the joint(s) connecting each component to the rest of the body.

Assumption (3) is necessary to provide a needed additional independent equation relating the component moments of inertia. The necessity for the introduction of this equation is established in Chapter 5.

### Background

Several researchers have attempted to obtain experimental values of the mass properties for parts of or entire humans. Much of the early work in this area consisted of obtaining centers of gravity and moments of inertia for parts of dissected frozen cadavers. Thus Braune and Fischer obtained these



values for a sample of three (Refs. 2, 3), Fischer later added another one (Ref. 8), and Dempster added eight more (Ref. 6). Barter used the data thus obtained to formulate a set of regression equations estimating the weight of each cadaver component as a function of total weight (Ref. 1).

Later experimental work centered on mass properties for living humans. Swearingen determined the centers of gravity for 5 living subjects in 67 body positions (Ref. 16). King investigated the locus of the center of gravity for various body positions (Ref. 12). Santschi, DuBois, and Omoto determined the center of gravity and moments of inertia for 66 subjects in various body positions by using a complex pendulum (Ref. 15).

Fowler mathematically showed how all the mass properties of a spacecraft could be experimentally determined in flight by using three independent, known thrusts (Ref. 9). This method is applicable to a human subject, and is the only available experimental means of determining products of inertia.

More recently, attempts have been made to model the human in order to predict mass properties as a function of body position. Hanavan (Ref. 11) assumed a simple geometric shape for each body component and derived formulas for inertia moments about component-fixed axes. DuBois and others improved Hanavan's model and added a space suit model consisting of hollow, simply shaped "shells" (Ref. 7). Tieber and Lindemuth (Ref. 17) made further modification of Hanavan's model and also modeled a Gemini space suit. They used an improved set of regression equations developed by Clauser and McConville (Ref. 4) to obtain estimates for component masses.

### Study Structure

This study first shows how whole-body mass properties can be determined from component mass properties. In this procedure component-fixed axes are

employed (Chapter 2). Then two methods for obtaining these component properties are presented. The first of these is a model formulated by Ernest P. Hanavan (Ref. 11), which assumes simple geometric shapes for the components (Chapter 3). The second method is a more general approach, making no assumption about component shapes except that they are rigid (Chapter 4). The necessity for the introduction of an additional equation relating the component moments in order to implement this latter method is then established (Chapter 5). This second method depends upon experimental data for whole-body mass properties for selected, fixed positions (Chapter 4, page 36), estimates of component masses (Chapter 6), and estimates of sums of component moments of inertia (Chapter 7).

A discussion of limitations of the second method and recommendations for further study are presented (Chapter 8), and finally a conclusion is given (Chapter 9).

## CHAPTER 2

### Determination of Whole-Body Mass Properties From Component Mass Properties with Respect to Component-Fixed Axes

#### Notation

The notation for this chapter and for the remainder of the study is as follows:

#### Scalar Notation

- $i, j$  - Subscripts denoting the quantity subscripted is for component number  $i$  or  $j$ , respectively.
- $m_B$  - the total body mass.
- $m_i$  - the mass of component  $i$ .
- $n$  - the total number of body components.
- $s_j$  - the sum of the moments of inertia of component  $j$  about an axis system centered at  $j$ 's center of mass.
- $\chi, \gamma, \mu$  - a set of rotation angles between the axes of  $C_i^1 - xyz$  and  $C_i - xyz$  (these angles are more explicitly defined later).
- $I, II, III, \dots, N$  - superscripts denoting the quantity superscripted is for a body position denoted by the Roman Numeral (  $N$  denotes the largest body position number).
- $\Delta^{I-II} ( )$  - an operator denoting the difference between the value of the quantity within the parentheses evaluated for position  $I$  and the value for position  $II$  (any Roman numerals can be used).

### Point and Coordinate System Notation

- A - an arbitrarily located (but known) point.
- A-xyz - an arbitrarily oriented Cartesian coordinate system centered at A .
- $C_B$  - the center of mass of the body.
- $C_i$  - the center of mass of component i .
- $C_i$ -xyz - a Cartesian coordinate system centered at  $C_i$  with axes respectively parallel to the axes of A-xyz .
- $C_i'$ -xyz - a component-fixed Cartesian coordinate system centered at  $C_i$ .
- $J_i$  - the joint (pivot point) at the end of component i closer to the body torso (e.g., the elbow is the forearm's  $J_i$  ).
- $J_i$ -xyz - a Cartesian coordinate system centered at  $J_i$  with axes respectively parallel to those of  $C_i$ -xyz. (and thus also parallel to those of A-xyz ).
- $J_i'$ -xyz - a Cartesian coordinate system centered at  $J_i$  with axes respectively parallel to those of  $C_i'$ -xyz.

### Vector Notation

- $\overline{b}_j^{I-II}$  - a 6x1 vector used for notational brevity (defined on page 30).
- $\overline{R}_B$  - the 3x1 vector of Cartesian coordinates of B relative to A-xyz.
- $\overline{R}_{C_i}$  - the 3x1 vector of Cartesian coordinates of  $C_i$  relative to A-xyz.

- $\bar{R}_{C_i/J_i}$  - the 3x1 vector of Cartesian coordinates of  $C_i$  relative to  $J_i$ -xyz.
- $\bar{R}_{C'_i/J_i}$  - the 3x1 vector of Cartesian coordinates of  $C_i$  relative to  $J'_i$ -xyz.
- $\bar{R}_{J_i}$  - the 3x1 vector of Cartesian coordinates of  $J_i$  relative to A-xyz.

The orientation (for a typical position) of some of the points, coordinate systems, and vectors appears in Figure 1 (next page).

#### Matrix Notation

- $A_j = \left[ \begin{array}{c} \Delta^{I-II}(T_j) \\ \Delta^{I-III}(T_j) \end{array} \right]$  - a 6x3 matrix used for notational brevity. ( $T_j$  is defined below).
- $I_B$  - the whole-body inertia tensor (3x3) about axes of A-xyz.
- $I_i$  - the inertia tensor of component  $i$  about the axes of A-xyz.
- $I_{i/C_i}$  - the inertia tensor of component  $i$  about the axes of  $C_i$ -xyz.
- $I_{i'}/C_i$  - the inertia tensor of component  $i$  about the axes of  $C'_i$ -xyz.
- $V_j^{I-II}$  - a 6x6 matrix that is a function of the elements of  $T_j$  (explicitly defined in Equation (4-8), page 30).
- $P_i$  - a 3x3 matrix which contains elements resulting from application of the parallel axis theorem. (more explicitly defined in

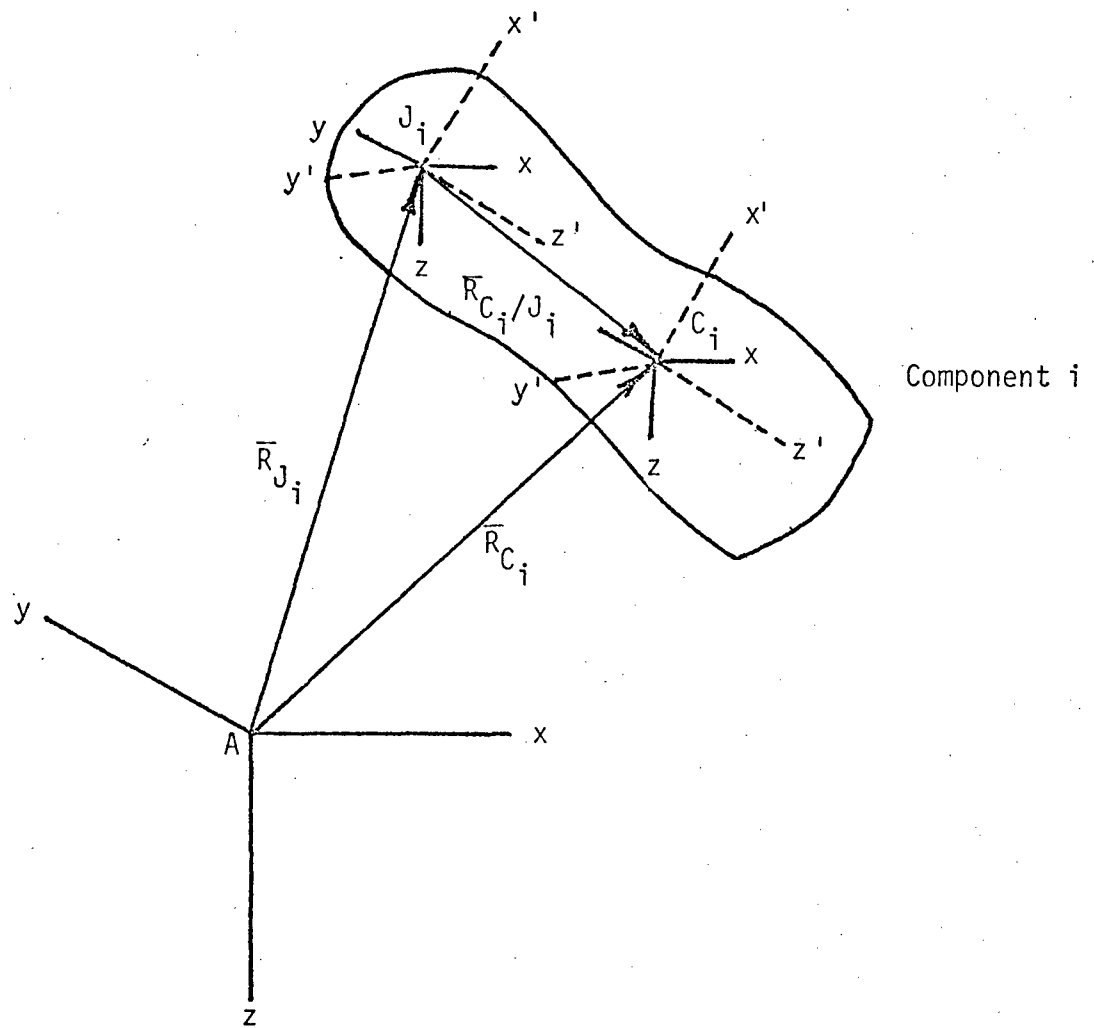


Figure 1. Vectors and Coordinate Systems

$$Q_j \equiv \left[ \begin{array}{c} \frac{v_j^{I-II}}{v_j^{I-III}} \end{array} \right] - \text{a } 12 \times 6 \text{ matrix used for notational brevity.}$$

$$Q_j^* \equiv \left[ \begin{array}{c} \frac{v_j^{I-II}}{v_j^{I-III}} \\ 111000 \end{array} \right] - \text{a } 13 \times 6 \text{ matrix used for notational brevity.}$$

### Analysis

The whole-body mass properties (with respect to A-xyz) can be expressed in terms of the component properties (with respect to A-xyz) as follows:

$$\text{Mass} \quad m_B = \sum_{i=1}^n m_i \quad (2-1)$$

$$\text{Center of Mass} \quad \bar{R}_B = \sum_{i=1}^n \left( \bar{R}_{C_i} \cdot m_i \right) / m_B \quad (2-2)$$

$$\text{Inertia Tensor} \quad I_B = \sum_{i=1}^n I_i \quad (2-3)$$

$$\text{But} \quad \bar{R}_{C_i} = \bar{R}_{C_i/J_i} + \bar{R}_{J_i} \quad (2-4)$$

$$\text{Also} \quad \bar{R}_{C_i/J_i} = T_i \bar{R}_{C_i'/J_i} \quad (2-5)$$

where the transformation matrix  $T_i$  is given by:

$$T_i = \begin{bmatrix} \cos\chi\cos\gamma & \cos\chi\sin\gamma\sin\mu - \sin\chi\cos\mu & \cos\chi\sin\gamma\cos\mu + \sin\chi\sin\mu \\ \sin\chi\cos\gamma & \sin\chi\sin\gamma\sin\mu + \cos\chi\cos\mu & \sin\chi\sin\gamma\cos\mu - \cos\chi\sin\mu \\ -\sin\gamma & \cos\gamma\sin\mu & \cos\gamma\cos\mu \end{bmatrix} \quad (2-6)$$

In the above matrix,  $\chi$ ,  $\gamma$ , and  $\mu$  are rotation angles illustrated in Figure 2. In the field of flight mechanics,  $\mu$ ,  $\chi$ , and  $\gamma$  correspond, respectively, to aircraft bank, heading, and flight path angles (Ref. 13, page 44). The orientation of the primed axes with respect to the unprimed can be described in terms of these three rotation angles. To define these rotations, two intermediate coordinate systems are introduced whose properties are as follows: the system  $Ox_1y_1z_1$  is obtained from the unprimed system by means of a rotation  $\chi$  around the  $z$ -axis; the system  $Ox_2y_2z_2$  is obtained from  $Ox_1y_1z_1$  by means of a rotation  $\gamma$  around the  $y_1$ -axis; the primed axes system is obtained from  $Ox_2y_2z_2$  by means of a rotation  $\mu$  around the  $x_2$ -axis.

Substituting Equation (2-5) into Equation (2-4) and then substituting (2-4) into (2-2), the whole-body center of mass becomes:

$$\bar{R}_B = \left[ \sum_{i=1}^n (T_i \bar{R}_{C_i} / J_i + \bar{R}_{J_i}) \cdot m_i \right] / m_B \quad (2-7)$$

By applying the parallel axis theorem, it is possible to express  $I_i$  in terms of  $I_{i/C_i}$  as follows:

$$I_i = I_{i/C_i} + P_i \quad i=1, \dots, n \quad (2-8)$$

where  $P_i$  is a 3x3 matrix whose elements are defined as follows:

Main Diagonal

Elements

$$P_i(p,p) = m_i \left[ R_{C_i}^2(q) + R_{C_i}^2(r) \right] \quad (2-9)$$

where  $p, q, r \in \{1,2,3\}$  and  $p \neq q \neq r$ .



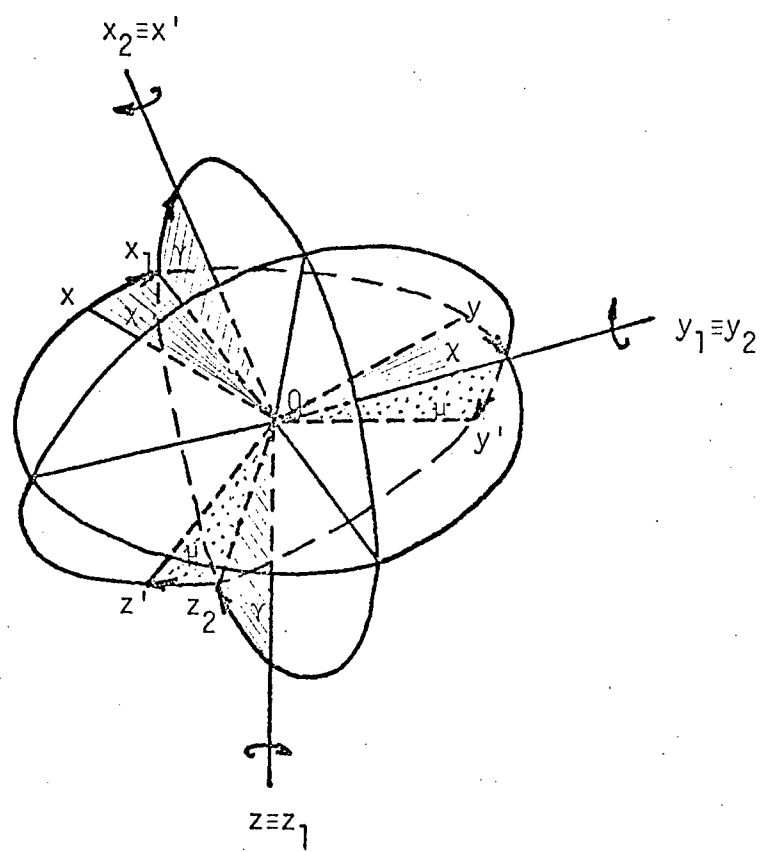


Figure 2. Rotation Angles

Off DiagonalElements

$$P_i(q,r) = m_i \left[ R_{C_i}(q) \cdot R_{C_i}(r) \right] \quad (2-10)$$

where  $q, r \in \{1,2,3\}$  and  $q \neq r$ .

Now the inertia tensor can be rotated by a similarity transformation, i.e.,  $I = T I' T^{-1}$ , where  $T$  is the transformation matrix from the primed to the unprimed axes systems (Ref. 10, page 370). Since our transformation is orthogonal, then  $T^{-1} = T^T$ , where the  $T$  superscript indicates the transpose (Ref. 14, page 207). Thus  $I = T I' T^T$ .

For  $I_{i/C_i}$ , then, we can write:

$$I_{i/C_i} = T_i I_{i'/C_i} T_i^T \quad (2-11)$$

Substituting (2-11) into (2-8), we obtain:

$$I_i = T_i I_{i'/C_i} T_i^T + P_i \quad (2-12)$$

Substituting (2-12) into (2-3) results in:

$$I_B = \sum_{i=1}^n \left( T_i I_{i'/C_i} T_i^T + P_i \right) \quad (2-13)$$

Note that in Equations (2-7) and (2-13), that while  $T_i (i=1, \dots, n)$  and  $P_i (i=1, \dots, n)$  are functions of position,  $R_{C_i/J_i} (i=1, \dots, n)$  and  $I_{i'/C_i} (i=1, \dots, n)$  are not (assuming components are rigid bodies). Equations (2-7) and (2-13), then, allow us to determine the whole-body mass properties for any position, given the body position and component mass properties with respect to component-fixed axes systems. A computer program capable of accomplishing this (PROGRAM WHOLE) is provided in Appendix A.

## CHAPTER 3

### The Hanavan Model

A personalized human body model was developed by Ernest P. Hanavan (Ref. 11), who obtained approximations for the  $R_{C_i/J_i}$  and  $I_{i'}/C_i$  ( $i=1, \dots, n$ ) by assuming simple geometric shapes for the body components. Figure 3 illustrates his model (from Ref. 11, page 8), and the assumed component shapes are indicated below:

- |       |                    |   |
|-------|--------------------|---|
| 1     | <u>Head</u>        | - right circular ellipsoid of revolution. |
| 2     | <u>Upper Torso</u> | - right elliptical cylinder.              |
| 3     | <u>Lower Torso</u> | - right elliptical cylinder.              |
| 4,5   | <u>Hands</u>       | - spheres.                                |
| 6,7   | <u>Upper Arms</u>  | - frustrums of right circular cones.      |
| 8,9   | <u>Forearms</u>    | - frustrums of right circular cones.      |
| 10,11 | <u>Upper Legs</u>  | - frustrums of right circular cones.      |
| 12,13 | <u>Lower Legs</u>  | - frustrums of right circular cones.      |
| 14,15 | <u>Feet</u>        | - frustrums of right circular cones.      |

Using 25 anthropometric body measurements of the person being modeled (Ref. 11, pages 9-10, A-1 through A-3) and Barter's regression equations for estimating masses of components (see Chapter 6), Hanavan defined his simply shaped components and provided formulas for component mass properties (Ref. 11, pages 13-24).

In this way, personalized approximations can be made for whole-body mass properties in any position. Hanavan compared his model's prediction with experimental data (Ref. 15), and claims his center of mass

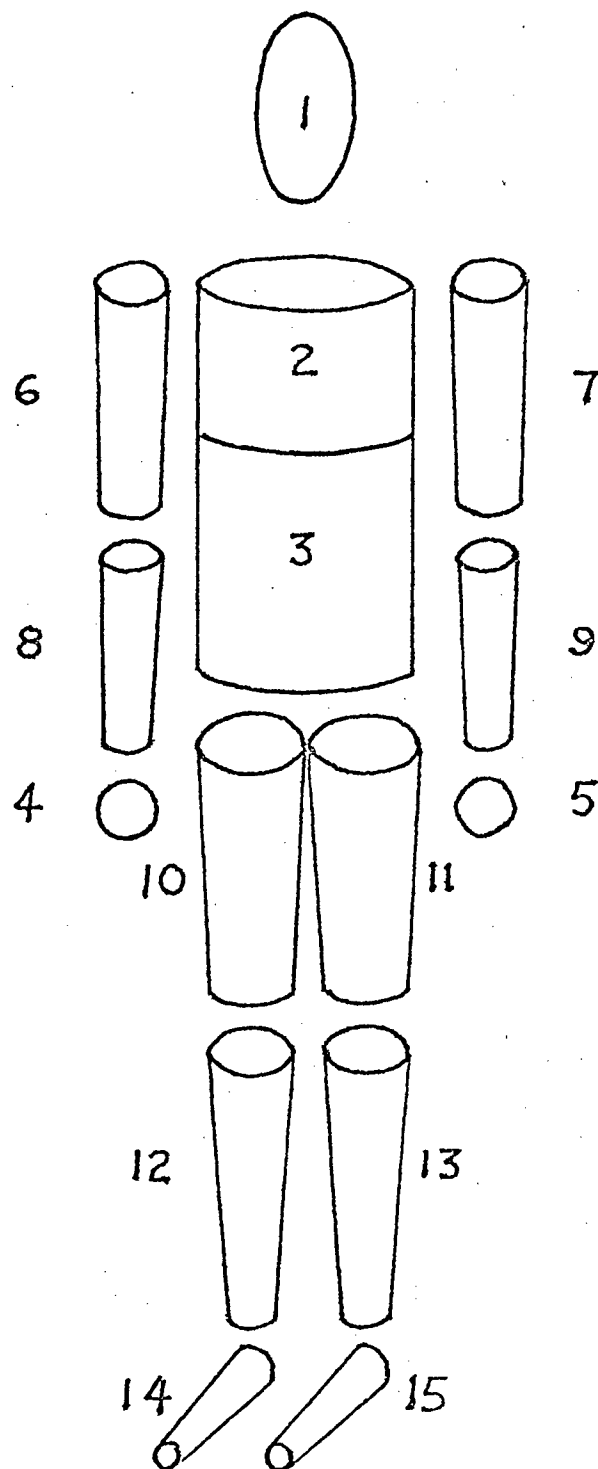


Figure 3. Hanavan's Model

is generally within 0.7 inches and moments of inertia are generally within 10% of experimental values from pendulum tests of live subjects. For preliminary design studies, this may be sufficiently accurate. Where better accuracy is needed, the method presented by the present study (Chapter 4) can be used.

The work of Hanavan was expanded by J. Dubois and others (Ref. 7) who added a model pressure suit consisting of "shells" of similar shapes surrounding each component. They also improved Hanavan's original model by assuming the torso hollow (thus taking air-filled lungs into account).

Further improvement of Hanavan's work was made by Tieber and Lindemuth (Ref. 17). Though the shape of Hanavan's components is not radically altered, pivot points are changed (Ref. 17, pages 16-21). Also, a different set of regression equations is used for estimating component masses (discussed in Chapter 6). A pressure suit model, similar to that of Dubois, is also provided (Ref. 17, pages 23-34), based on a G-4c suit (used in the Gemini project). Tieber and Lindemuth claimed that the center of gravity errors are generally reduced to less than 0.3 inches, but the moment of inertia errors are generally as high as 15% of experimental values.

The present study presents a more general method for obtaining  $\bar{C}_i/J_i$  and  $I_i/C_i$  ( $i=1, \dots, n$ ) than Hanavan's model (Chapter 4). But this method, it will be seen, depends on obtaining estimates for the sums of component inertia moments. Hanavan's model provides one possible method of obtaining these estimates (see Chapter 7).

This more general method will also depend on estimates of component masses. Barter's regression equations will be presented in Chapter 6 as a possible estimation method.

## CHAPTER 4

### Determination of Component Mass Properties About Component-Fixed Axes

#### Overview

As was shown in Chapter 2, whole-body mass properties for any position can be found if component mass properties about component-fixed axes are known. Hanavan's model of the previous chapter obtained these by assuming simple geometric shapes for the components. This chapter provides a more general method which assumes nothing about the component shapes, only that the components are rigid bodies. It does depend on the assumption that the following information can be obtained:

- (1) Experimental data for the whole-body center of mass and the inertia tensor for a living human (discussed later in this chapter).
- (2) An estimate for the mass of each of the various components of the body (defined by the researcher). Chapter 6 presents two methods, one using either of the two sets of regression equations referred to in Chapter 3, the other using experimental data for casts of the particular subject's body parts.
- (3) An estimate for the sum of the three moments of inertia for each body component. Chapter 7 provides two methods, one using Hanavan's simply shaped components, the other using experimental data for casts of the subject's body parts.

The basis for the solution for component mass properties is the formation of what will be called mass property difference equations. These are formed by subtracting the equations for whole-body mass properties (Equations (2-7)

and (2-13)) for selected body positions. Using the center of mass difference equation and estimates for component masses, component centers of mass can be determined. Using the inertia tensor difference equation, estimates for component masses, and estimates for sums of component moments of inertia, component inertia tensors can be determined. It will be shown in Chapter 5 that estimates for sums of component moments of inertia provide a needed additional independent equation relating elements of component inertia tensors.

### Component Centers of Mass

For two different body positions, say I and II, identical except for the position of one component, say number j, Equation (2-7) becomes:

$$\bar{R}_B^I = \left[ \sum_{i=1}^n (T_i^I \bar{R}_{C_i^I} / J_i + \bar{R}_{J_i^I}) \cdot m_i \right] / m_B$$

$$\bar{R}_B^{II} = \left[ \sum_{i=1}^n (T_j^{II} \bar{R}_{C_i^I} / J_i + \bar{R}_{J_i^{II}}) \cdot m_i \right] / m_B$$

Splitting up the summation and subtracting these two equations:

$$\bar{R}_B^I - \bar{R}_B^{II} = \left[ \sum_{i=1}^n T_i^I \bar{R}_{C_i^I} / J_i m_i - \sum_{i=1}^n T_i^{II} \bar{R}_{C_i^I} / J_i m_i + \sum_{i=1}^n \bar{R}_{J_i^I} m_i - \sum_{i=1}^n \bar{R}_{J_i^{II}} m_i \right] / m_B$$

Equivalently:

$$m_B \cdot \Delta^{I-II}(\bar{R}_B) = \sum_{i=1}^n \left[ \Delta^{I-II}(T_i) \right] \bar{R}_{C_i^I} / J_i m_i + \left[ \Delta^{I-II}(\bar{R}_{J_i}) \right] m_i \quad (4-1)$$

But I and II are identical positions except for the position of component j. Thus:

$$T_i^I = T_i^{II} \quad i=1, \dots, n \quad (i \neq j)$$

$$R_{J_i}^I = R_{J_i}^{II} \quad i=1, \dots, n$$

Hence Equation (4-1) becomes:

$$m_B \left[ \Delta^{I-II}(\bar{R}_B) \right] = \left[ \Delta^{I-II}(T_j) \right] \bar{R}_{C_j/J_j} m_j$$

Equivalently:

$$\left[ \Delta^{I-II}(T_j) \right] \bar{R}_{C_j/J_j} = (m_B/m_j) \left[ \Delta^{I-II}(\bar{R}_B) \right] \quad (4-2)$$

In the above equation  $\bar{R}_B^I$  and  $\bar{R}_B^{II}$  can be determined by experiment (see page ), as can  $m_B$ . The quantity  $m_j$  can be estimated (see page ). The matrices  $T_j^I$  and  $T_j^{II}$  represent arbitrarily selected, known positions. Thus Equation (4-2) represents a system of three linear equations in three unknowns, the elements of  $\bar{R}_{C_j/J_j}$ . Clearly, if  $\Delta^{I-II}(T_j)$  is nonsingular,  $\bar{R}_{C_j/J_j}$  can be solved for directly in Equation (4-2). However, the following analysis proves that  $\Delta^{I-II}(T_j)$  is singular for all choices of positions I and II.

The orientation of the component-fixed axes  $C_j^I$ -xyz is arbitrary. Thus, without loss of generality,  $C_j^I$ -xyz can be oriented such that  $\chi^I = 0^\circ$ ,  $\gamma^I = 0^\circ$ , and  $\mu^I = 0^\circ$  for position I, for which:

$$T_j^I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For any other position, II:



$$T_j^{II} = \begin{bmatrix} \cos X^{II} \cos Y^{II} & \cos X^{II} \sin Y^{II} \sin \mu^{II} & \cos X^{II} \sin Y^{II} \cos \mu^{II} \\ -\sin X^{II} \cos \mu^{II} & +\sin X^{II} \sin \mu^{II} & \\ \sin X^{II} \cos Y^{II} & \sin X^{II} \sin Y^{II} \sin \mu^{II} & \sin X^{II} \sin Y^{II} \cos \mu^{II} \\ +\cos X^{II} \cos \mu^{II} & -\cos X^{II} \sin \mu^{II} & \\ -\sin Y^{II} & \cos Y^{II} \sin \mu^{II} & \cos Y^{II} \cos \mu^{II} \end{bmatrix}$$

Hence:

$$\Delta^{I-II}(T_j) = \begin{bmatrix} 1 - \cos X^{II} \cos Y^{II} & -\cos X^{II} \sin Y^{II} \sin \mu^{II} & -\cos X^{II} \sin Y^{II} \cos \mu^{II} \\ +\sin X^{II} \cos \mu^{II} & -\sin X^{II} \sin \mu^{II} & \\ -\sin X^{II} \cos Y^{II} & 1 - \sin X^{II} \sin Y^{II} \sin \mu^{II} & -\sin X^{II} \sin Y^{II} \cos \mu^{II} \\ -\cos X^{II} \cos \mu^{II} & \cos X^{II} \sin \mu^{II} & \\ \sin Y^{II} & -\cos Y^{II} \sin \mu^{II} & 1 - \cos Y^{II} \cos \mu^{II} \end{bmatrix}$$

The determinant of the above matrix turns out to be zero. This is seen by expanding the determinant and substituting appropriate trigonometric identities. Consequently, for any selection of positions I and II, the matrix  $\Delta^{I-II}(T_j)$  is singular.

Hence Equation (4-2) represents a dependent system of equations (Ref. 14, page 133), and therefore does not uniquely determine the elements of  $\bar{R}_{C_j/J_j}$ . However, if an arbitrary third position (III) is introduced, the following difference equation results, identical in form to Equation (4-2):

$$\left[ \Delta^{I-III}(T_j) \right] \bar{R}_{C_j/J_j} = (m_B/m_j) \left[ \Delta^{I-III}(\bar{R}_B) \right] \quad (4-3)$$

The combination of Equations (4-2) and (4-3) constitute a system of equations in three unknowns. For judicious selection of positions I,

II, and III, it turns out that this system can contain three independent equations, thus determining a unique solution for  $\bar{R}_{C_j/J_j}$ . However, it may be inconvenient to isolate three independent equations. An alternative solution of the system which does not require this isolation does exist, and its derivation follows.

Combining Equations (4-2) and (4-3) into one equation:

$$\begin{bmatrix} \frac{\Delta^{I-II}(T_j)}{\Delta^{I-III}(T_j)} \end{bmatrix} \bar{R}_{C_j/J_j} = \left( m_B/m_j \right) \begin{bmatrix} \frac{\Delta^{I-II}(\bar{R}_B)}{\Delta^{I-III}(\bar{R}_B)} \end{bmatrix}$$

Defining the matrix  $A_j = \begin{bmatrix} \frac{\Delta^{I-II}(T_j)}{\Delta^{I-III}(T_j)} \end{bmatrix}$  and multiplying the above equation by  $A_j^T$ :

$$A_j^T A_j \bar{R}_{C_j/J_j} = (m_B/m_j) A_j^T \begin{bmatrix} \frac{\Delta^{I-II}(\bar{R}_B)}{\Delta^{I-III}(\bar{R}_B)} \end{bmatrix} \quad (4-4)$$

But for any matrix  $A_j$ ,  $A_j^T A_j$  and  $A_j$  have the same rank, or the number of independent rows (or columns) (Ref. 14, page 139). Thus if  $A_j$  has rank 3, then so has  $A_j^T A_j$ . But  $A_j^T A_j$  is  $3 \times 3$ ; hence,  $A_j^T A_j$  must have an inverse if it has rank 3. Therefore, if  $A_j$  is of rank 3, then Equation (4-4) can be solved directly for  $\bar{R}_{C_j/J_j}$  as follows:

$$\bar{R}_{C_j/J_j} = (m_B/m_j) (A_j^T A_j)^{-1} A_j^T \begin{bmatrix} \frac{\Delta^{I-II}(\bar{R}_B)}{\Delta^{I-III}(\bar{R}_B)} \end{bmatrix} \quad (4-5)$$

Whether or not  $A_j$  has rank 3 depends upon the selection of positions I, II, and III. Two specific examples follow, the first of which is an  $A_j$  of rank 3, the second of which is an  $A_j$  of rank 2, thus substantiating the earlier statement that the selection of body positions must be judicious.

Example 1

Let positions I, II, III have rotation angles as indicated below.

POSITION	x	y	u
I	0	0	0
II	0	$\sin^{-1}(3/5)$	0
III	0	0	$\sin^{-1}(3/5)$

For this selection of positions:

$$T_j^I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_j^{II} = \begin{bmatrix} 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ -3/5 & 0 & 4/5 \end{bmatrix} \quad T_j^{III} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix}$$

Thus:

$$A_j = \begin{bmatrix} \Delta^{I-II}(T_j) \\ \Delta^{I-III}(T_j) \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & -3/5 \\ 0 & 0 & 0 \\ 3/5 & 0 & 1/5 \\ 0 & 0 & 0 \\ 0 & 1/5 & 3/5 \\ 0 & -3/5 & 1/5 \end{bmatrix}$$

For the above matrix  $A_j$ :

$$A_j^T A_j = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}$$

which is clearly of full rank. Thus  $(A_j^T A_j)^{-1}$  exists and is in fact as follows:

$$(A_j^T A_j)^{-1} = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 1.25 \end{bmatrix}$$

Therefore, for this selection of body positions Equation (4-7) will provide a unique solution for the elements of  $\bar{R}_{C_j'/J_j}$ . In Example 2, which follows, this will not be the case.

Example 2

For this example let positions I, II, III have rotation angles as indicated below.

POSITION	X	Y	u
I	0	0	0
II	0	$\sin^{-1}(3/5)$	0
III	0	$\sin^{-1}(4/5)$	0

For this selection of positions:

$$T_j^I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad T_j^{II} = \begin{bmatrix} 4/5 & 0 & 3/5 \\ 0 & 1 & 0 \\ -3/5 & 0 & 4/5 \end{bmatrix} \quad T_j^{III} = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & 3/5 \end{bmatrix}$$

Thus:

$$A_j = \begin{bmatrix} \Delta^{I-II}(T_j) \\ \Delta^{I-III}(T_j) \end{bmatrix} = \begin{bmatrix} 1/5 & 0 & -3/5 \\ 0 & 0 & 0 \\ 3/5 & 0 & 1/5 \\ 2/5 & 0 & -4/5 \\ 0 & 0 & 0 \\ 4/5 & 0 & 2/5 \end{bmatrix}$$

For the above matrix A:

$$A_j^T A_j = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}$$

which is clearly of rank 2. Therefore,  $A_j^T A_j$  will not have an inverse. The matrix  $A_j$  provides only two independent equations and thus does not determine a unique value for the elements of  $\bar{R}_{C_j'/J_j}$ .

An apparent (though not proved) general rule for the selection of body positions to insure a unique value for  $\bar{R}_{C_j'/J_j}$  is to choose positions such that  $\bar{R}_{C_i/J_i}^I$ ,  $\bar{R}_{C_i/J_i}^{II}$ , and  $\bar{R}_{C_i/J_i}^{III}$  are not all three coplanar. Thus, in Example 1, where these three vectors did not lie in the same plane, a unique value for  $\bar{R}_{C_j'/J_j}$  was determined. But for Example 2, where the vectors were coplanar, a unique value was not determined.

In the analysis of this section, it was tacitly assumed that for component  $j$ , it is possible to define positions I, II, and III such that only  $j$  is different. But this is not possible for some components, for example if the thigh position is different, then so must the calf and foot positions be different. More generally, Equation (4-5) is applicable directly only to those components connected to only one other component, hereafter called external components. Those components which are connected to two other components shall be called internal components. Note that the central component from which the limbs emanate fit into neither of these categories of components, as more than two components are connected to it. For convenience, this central component will be numbered  $n$ . Typically, of course, this central component is simply the torso, but the researcher has the option to include the head and neck as part of the central component if they are assumed to be immobile. This assumption may be quite reasonable for a pressure-suited astronaut, with restricted head mobility.

A method for determining the center of mass for any internal component and, finally, one component external to it is now presented. Using the method just presented, the center of mass of the external component can be

determined. For notational convenience let this external joint be numbered  $j+1$  and the internal component whose center of mass we seek be numbered  $j$ . Then difference equations similar to Equations (4-2) and (4-3) can be formed but will have an extra term (because the contribution of component  $j+1$  does not subtract out) and will be as follows:

$$\begin{aligned} \left[ \Delta^{I-II}(T_j) \right] \bar{R}_{C'_j/J_j} &= \left( m_B/m_j \right) \left[ \Delta^{I-II}(\bar{R}_B) \right] \\ &- \left( m_{j+1}/m_j \right) \left[ \Delta^{I-II}(T_{j+1}) \bar{R}_{C'_{j+1}/J_{j+1}} + \Delta^{I-II}(\bar{R}_{J_{j+1}}) \right] \end{aligned}$$

and

$$\begin{aligned} \left[ \Delta^{I-III}(T_j) \right] \bar{R}_{C'_j/J_j} &= \left( m_B/m_j \right) \left[ \Delta^{I-III}(\bar{R}_B) \right] \\ &- \left( m_{j+1}/m_j \right) \left[ \Delta^{I-III}(T_{j+1}) \bar{R}_{C'_{j+1}/J_{j+1}} + \Delta^{I-III}(\bar{R}_{J_{j+1}}) \right] \end{aligned}$$

In the above equations, the center of mass of the external component,  $\bar{R}_{C'_{j+1}/J_{j+1}}$  must be predetermined. The  $T_j$ ,  $T_{j+1}$ , and  $\bar{R}_{J_{j+1}}$  are functions of the body positions I, II, and III. Combining the above equations, an equation analagous to Equation (4-5) is obtained and is as follows:

$$\begin{aligned} \bar{R}_{C'_j/J_j} &= \left( A_j^T A_j \right)^{-1} A_j^T \left[ \left( m_B/m_j \right) \left[ \begin{array}{c} \Delta^{I-II}(\bar{R}_B) \\ \Delta^{I-III}(\bar{R}_B) \end{array} \right] \right. \\ &\quad \left. - \left( m_{j+1}/m_j \right) \left\{ \left[ \begin{array}{c} \Delta^{I-II}(T_{j+1}) \\ \Delta^{I-III}(T_{j+1}) \end{array} \right] \bar{R}_{C'_{j+1}/J_{j+1}} + \left[ \begin{array}{c} \Delta^{I-II}(\bar{R}_{J_{j+1}}) \\ \Delta^{I-III}(\bar{R}_{J_{j+1}}) \end{array} \right] \right\} \right] \end{aligned}$$

A similar analysis can be carried out to find the center of mass for each internal component with more than one external component. The following equation results:

$$\bar{R}_{C'_j/J_j} = (A_j^T A_j)^{-1} A_j^T \left[ \left( m_B/m_j \right) \left[ \frac{\Delta^{I-II}(\bar{R}_B)}{\Delta^{I-III}(\bar{R}_B)} \right] - \sum_{\substack{i \text{ external} \\ \text{to } j}} \left( m_i/m_j \right) \left\{ \left[ \frac{\Delta^{I-II}(\tau_i)}{\Delta^{I-III}(\tau_i)} \right] \bar{R}_{C'_i/J_i} + \left[ \frac{\Delta^{I-II}(\bar{R}_{J_i})}{\Delta^{I-III}(\bar{R}_{J_i})} \right] \right\} \right]$$

The above equation can actually be considered generally applicable to any component except the central component if the summations are ignored for external components (for which there are no  $i$  external to  $j$ ). Assuming  $\bar{R}_{C'_i/J_i}$  ( $i=1, \dots, n-1$ ) have already been determined, the center of mass of the central component is given by a simple manipulation of Equation (2-7) for any position:

$$\bar{R}_{C'_n/J_n} = T_n^T \left[ (m_B/m_n) \bar{R}_B - \sum_{i=1}^{n-1} \left( \frac{m_i}{m_n} \right) \left( T_i \bar{R}_{C'_i/J_i} + \bar{R}_{J_i} \right) \right] - \bar{R}_{J_n}$$

$J_n$  in the above equation can be any conveniently defined fixed point to the central component.

Using the equations developed, then, all component centers of mass can be found.

#### Component Inertia Tensors

For two different body positions I and II, identical except for the position of one external component  $j$ , Equation (2-13) becomes:

$$I_B^I = \sum_{i=1}^n \left\{ T_i^I I_{i/C_i} (T_i^I)^T + P_i^I \right\}$$

$$I_B^{II} = \sum_{i=1}^n \left\{ T_i^{II} I_{i/C_i} (T_i^{II})^T + P_i^{II} \right\}$$

Subtracting these two equations:

$$\Delta^{I-II}(I_B) = I_B^I - I_B^{II} = \sum_{i=1}^n \left\{ T_i^I I_{i/C_i} (T_i^I)^T - T_i^{II} I_{i/C_i} (T_i^{II})^T + P_i^I - P_i^{II} \right\}$$

But for all components except  $j$ ,

$$T_i^I = T_i^{II} \quad ; \quad i=1, \dots, n \quad (i \neq j)$$

$$P_i^I = P_i^{II} \quad ; \quad i=1, \dots, n \quad (i \neq j)$$

Substituting, the following equation is obtained:

$$\Delta^{I-II}(I_B) = T_j^I I_{j/C_j} (T_j^I)^T - T_j^{II} I_{j/C_j} (T_j^{II})^T + P_j^I - P_j^{II}$$

Equivalently:

$$\Delta^{I-II}(I_B) - \Delta^{I-II}(P_j) = T_j^I I_{j/C_j} (T_j^I)^T - T_j^{II} I_{j/C_j} (T_j^{II})^T \quad (4-6)$$

In this last equation,  $I_B^I$  and  $I_B^{II}$  are obtainable from whole-body experimental data (see Chapter 6). As can be seen in Equations (2-9) and (4-10),  $P_j$  is a function of  $m_j$ ,  $T_j$ ,  $\bar{R}_{C_j/J_j}$ , and  $\bar{R}_{J_j}$ . The choice of body positions determines  $T_j$  and  $\bar{R}_{J_j/A}$ . The quantity  $m_j$  is obtainable therefore from the methods presented in Chapter 6. The quantity  $\bar{R}_{C_j/J_j}$  is obtainable from the method presented in the previous section. Thus  $P_j$  can be found for any positions I and II, and Equation (4-6) becomes a matrix equation in one unknown matrix,  $I_{j/C_j}$ . However, there is no obvious way to solve (4-6) directly for  $I_{j/C_j}$ . Hence it becomes necessary to express the



matrices in terms of elements, carry out the operations, and equate elements. In this way Equation (4-6) is equivalent to the following system of nine linear equations in nine unknowns (the elements of  $I_j'/C_j$ ):

#### Main Diagonal Elements

$$\Delta^{I-II}(I_B(p,p)) - \Delta^{I-II}(P_j(p,p)) = \sum_{\ell=1}^3 \sum_{k=1}^3 [\Delta^{I-II}(T_j(q,k)T_j(r,\ell))] I_j'/C_j(k,\ell)$$

where  $p, q, r \in \{1, 2, 3\}$  ( $p \neq q \neq r$ )

#### Off Diagonal Elements

$$\Delta^{I-II}(I_B(q,r)) - \Delta^{I-II}(P_j(q,r)) = \sum_{\ell=1}^3 \sum_{k=1}^3 [\Delta^{I-II}(T_j(q,k)T_j(r,\ell))] I_j'/C_j(k,\ell)$$

where  $q, r \in \{1, 2, 3\}$  ,  $q \neq r$

Because of the symmetry of the inertia tensor, the above system can be reduced to a system of six equations in six unknowns (the six independent elements of  $I_j'/C_j$ ). The off diagonal equations given above actually provide only three rather than six equations. For example if  $q=1$ ,  $r=2$ , then above equation becomes:

$$\Delta^{I-II}(I_B(1,2)) - \Delta^{I-II}(P_j(1,2)) = \sum_{\ell=1}^3 \sum_{k=1}^3 [\Delta(T_j(1,k)T_j(2,\ell))] I_j'/C_j(k,\ell)$$

But for  $q=2$ ,  $r=1$  :

$$\Delta^{I-II}(I_B(2,1)) - \Delta^{I-II}(P_j(2,1)) = \sum_{\ell=1}^3 \sum_{k=1}^3 [\Delta(T_j(2,k)T_j(1,\ell))] I_j'/C_j(k,\ell)$$

but  $I_B^I$  ,  $I_B^{II}$  ,  $P_j^I$  ,  $P_j^{II}$  , and  $I_j'/C_j$  are symmetric matrices, or:

$$I_B^I(2,1) = I_B^I(1,2)$$

$$I_B^{II}(2,1) = I_B^{II}(1,2)$$

$$P_j^I(2,1) = P_j^I(1,2)$$

$$P_j^{II}(2,1) = P_j^{II}(1,2)$$

$$I_{j/C_j}'(k,\ell) = I_{j/C_j}'(\ell,k)$$

Substituting these into the earlier equation for  $q=2$ ,  $r=1$ , reversing the order of summation, and commuting the multiplications within the summation:

$$\Delta^{I-II}(I_B(1,2)) - \Delta^{I-II}(P_j(1,2)) = \sum_{k=1}^3 \sum_{\ell=1}^3 \left[ \Delta^{I-II}(T_j(2,k)T_j(1,\ell)) \right] I_{j/C_j}'(\ell,k)$$

But  $k$  and  $\ell$  are dummy variables. Hence the equation above for  $q=2$ ,  $r=1$  is seen to be identical to the earlier presented equation for  $q=1$ ,  $r=2$ . Similarly, identical equations result for  $q=1$ ,  $r=3$ , and for  $q=3$ ,  $r=1$ . The same is true for the cases  $q=2$ ,  $r=3$  and  $q=3$ ,  $r=2$ .

In this way the original system of nine equations reduces to the following linear system of six equations in six unknowns (the independent elements  $I_{j/C_j}'$ ):

#### Main Diagonal Elements

$$\begin{aligned} \Delta^{I-II}(I_B(p,p)) - \Delta^{I-II}(P_j(p,p)) &= \sum_{k=1}^3 \left[ \Delta^{I-II}(T_j(p,k)^2) \right] I_{j/C_j}'(p,p) \\ &+ 2 \sum_{\ell=1}^3 \sum_{\substack{k=1 \\ k > \ell}}^3 \left[ \Delta(T_j(q,k)T_j(r,k)) \right] I_{j/C_j}'(\ell,k) \end{aligned}$$

where  $p, q, r \in \{1, 2, 3\}$ ,  $p \neq q \neq r$   $q < r$

### Off Diagonal Elements

$$\begin{aligned} \Delta^{I-II}(I_B(q, r)) - \Delta^{I-II}(P_i(q, r)) &= \sum_{k=1}^3 \left[ \Delta^{I-II}(T_j(q, k)T_j(r, k)) \right] I_{j'/C_j}(p, p) \\ &+ \sum_{\ell=1}^3 \sum_{\substack{k=1 \\ k > \ell}}^3 \left[ \Delta^{I-II}(T_j(q, k)T_j(r, \ell) \right. \\ &\quad \left. + T_j(q, \ell)T_j(r, k)) \right] I_{j'/C_j}(\ell, k) \end{aligned}$$

where  $q, r \in \{1, 2, 3\}$ ,  $q < r$

In matrix form this linear system can be written as follows:

$$V_j^{I-II} \begin{bmatrix} I_{j'/C_j}(1, 1) \\ I_{j'/C_j}(2, 2) \\ I_{j'/C_j}(3, 3) \\ I_{j'/C_j}(1, 2) \\ I_{j'/C_j}(1, 3) \\ I_{j'/C_j}(2, 3) \end{bmatrix} = \bar{b}_j^{I-II}$$

(4-7)

where

$$\bar{b}_j^{I-II} \equiv \begin{bmatrix} \Delta^{I-II} (I_B (1,1)) - \Delta^{I-II} (P_j(1,1)) \\ \Delta^{I-II} (I_B (2,2)) - \Delta^{I-II} (P_j(2,2)) \\ \Delta^{I-II} (I_B (3,3)) - \Delta^{I-II} (P_j(3,3)) \\ \Delta^{I-II} (I_B (1,2)) - \Delta^{I-II} (P_j(1,2)) \\ \Delta^{I-II} (I_B (1,3)) - \Delta^{I-II} (P_j(1,3)) \\ \Delta^{I-II} (I_B (2,3)) - \Delta^{I-II} (P_j(2,3)) \end{bmatrix}$$

and

$$V_j^{I-II} \equiv$$

$$\begin{bmatrix} \Delta(T_j(1,1)^2) & \Delta(T_j(1,2)^2) & \Delta(T_j(1,3)^2) & 2(\Delta T_j(1,1)T_j(1,2)) & 2(\Delta T_j(1,1)T_j(1,3)) & 2(\Delta T_j(1,2)T_j(1,3)) \\ \Delta(T_j(2,1)^2) & \Delta(T_j(2,2)^2) & \Delta(T_j(2,3)^2) & 2(\Delta T_j(2,1)T_j(2,2)) & 2(\Delta T_j(2,1)T_j(2,3)) & 2(\Delta T_j(2,2)T_j(2,3)) \\ \Delta(T_j(3,1)^2) & \Delta(T_j(3,2)^2) & \Delta(T_j(3,3)^2) & 2(\Delta T_j(3,1)T_j(3,2)) & 2(\Delta T_j(3,1)T_j(3,3)) & 2(\Delta T_j(3,2)T_j(3,3)) \\ \Delta(T_j(2,1)T_j(3,1)) & \Delta(T_j(2,2)T_j(3,2)) & \Delta(T_j(2,3)T_j(3,3)) & [\Delta(T_j(2,2)T_j(3,1))] & [\Delta(T_j(2,3)T_j(3,1))] & [\Delta(T_j(2,3)T_j(3,2))] \\ & & & +\Delta(T_j(2,1)T_j(3,2)) & +\Delta(T_j(2,1)T_j(3,3)) & +\Delta(T_j(2,2)T_j(3,3))] \\ \Delta(T_j(1,1)T_j(3,1)) & \Delta(T_j(1,2)T_j(3,2)) & \Delta(T_j(1,3)T_j(3,3)) & [\Delta(T_j(1,2)T_j(3,1))] & [\Delta(T_j(1,3)T_j(3,1))] & [\Delta(T_j(1,3)T_j(3,2))] \\ & & & +\Delta(T_j(1,1)T_j(3,2)) & +\Delta(T_j(1,1)T_j(3,3)) & +\Delta(T_j(1,2)T_j(3,3))] \\ \Delta(T_j(1,1)T_j(2,1)) & \Delta(T_j(1,2)T_j(2,2)) & \Delta(T_j(1,3)T_j(2,3)) & [\Delta(T_j(1,2)T_j(2,1))] & [\Delta(T_j(1,3)T_j(2,1))] & [\Delta(T_j(1,3)T_j(2,2))] \\ & & & +\Delta(T_j(1,1)T_j(2,2)) & +\Delta(T_j(1,1)T_j(2,3)) & +\Delta(T_j(1,2)T_j(2,3))] \end{bmatrix}^{I-II}$$

(4-8)

Clearly, if the 6 x 6 matrix  $V_j^{I-II}$  has an inverse, then the vector of the elements of  $I_j'/C_j$  can be solved for directly in Equation (4-7). However, it turns out that  $V_j^{I-II}$  is singular for any selection of positions (rigorously proved in Chapter 5). In fact, for every one of numerous examples, the rank of  $V_j^{I-II}$  is only 4, even when position II's are all different than position I's.

If a third position (III), identical to I and II except for the position

of component  $j$ , is introduced, then an equation like (4-7) can be written, as follows:

$$v_j^{I-III} \begin{bmatrix} I_{j/C_j}'(1,1) \\ I_{j/C_j}'(2,2) \\ I_{j/C_j}'(3,3) \\ I_{j/C_j}'(1,2) \\ I_{j/C_j}'(1,3) \\ I_{j/C_j}'(2,3) \end{bmatrix} = \bar{b}_j^{I-III} \quad (4-9)$$

where  $v_j^{I-III}$  and  $\bar{b}_j^{I-III}$  have the same form as the previously given equations for  $v_j^{I-II}$  and  $\bar{b}_j^{I-II}$ , respectively.

Combining Equations (4-7) and (4-9):

$$\begin{bmatrix} v_j^{I-II} \\ \hline v_j^{I-III} \end{bmatrix} \begin{bmatrix} I_{j/C_j}'(1,1) \\ (2,2) \\ (3,3) \\ (1,2) \\ (1,3) \\ (2,3) \end{bmatrix} = \begin{bmatrix} \bar{b}_j^{I-II} \\ \hline \bar{b}_j^{I-III} \end{bmatrix} \quad (4-10)$$

For notational brevity, the  $12 \times 6$  matrix  $\begin{bmatrix} v_j^{I-II} \\ \hline v_j^{I-III} \end{bmatrix}$  shall be denoted by the symbol  $Q_j$ . Now, if  $Q_j$  be of rank 6, then so will the  $6 \times 6$  matrix  $(Q_j^T Q_j)^{-1}$  will exist, and Equation (4-10) can be solved directly for the vector of the elements of  $I_{j/C_j}'$ . However it turns out that

the rank of  $Q_j$  is less than 6, the actual rank depending on the selection of positions I, II, and III. That the rank of  $Q_j$  is less than six is proved in Chapter 5, where it is additionally proved that no matter how many positions are incorporated into Equation (4-10), the  $Q_j$  matrix will still be of rank less than 6.

For three positions, it is possible to find specific cases for which the rank of  $Q_j$  is 5. Thus there is hope for a unique solution if one additional equation in the elements of  $I_{j/C_j}'$  is introduced. An approximation for the sum of the moments of inertia of component  $j$  provides this equation:

$$I_{j/C_j}'(1,1) + I_{j/C_j}'(2,2) + I_{j/C_j}'(3,3) = s_j \quad (4-11)$$

The constant  $s_j$  can be approximated using the methods presented in Chapter 7.

Incorporating Equation (4-11) into (4-10):

$$Q_j^* \begin{bmatrix} I_{j/C_j}'(1,1) \\ I_{j/C_j}'(2,2) \\ I_{j/C_j}'(3,3) \\ I_{j/C_j}'(1,2) \\ I_{j/C_j}'(1,3) \\ I_{j/C_j}'(2,3) \end{bmatrix} = \begin{bmatrix} \overline{b_j} I-II \\ \overline{b_j} I-III \\ s_j \end{bmatrix} \quad (4-12)$$

$$Q_j^* = \begin{bmatrix} v_j^{I-II} \\ \hline v_j^{I-III} \\ \hline 1 \ 1 \ 1 \ 0 \ 0 \ 0 \end{bmatrix}.$$

Now if  $Q_j^*$  is of rank 6, then  $(Q_j^{*T} Q_j^*)^{-1}$  exists and Equation (4-12) can be solved for the elements of  $I_{j/C_j}'$  as follows:

$$\begin{bmatrix} I_{j/C_j}'(1,1) \\ I_{j/C_j}'(2,2) \\ I_{j/C_j}'(3,3) \\ I_{j/C_j}'(1,2) \\ I_{j/C_j}'(1,3) \\ I_{j/C_j}'(2,3) \end{bmatrix} = (Q_j^{*T} Q_j^*)^{-1} Q_j^{*T} \begin{bmatrix} \bar{b}_j^{I-II} \\ \hline \bar{b}_j^{I-III} \\ \hline s_j \end{bmatrix} \quad (4-13)$$

An example of body positions I, II, and III for which (4-13) yields a unique solution for the elements of  $I_{j/C_j}'$  follows.

### Example 3

Let positions I, II, and III be defined the same as they were for Example 1. For these positions,

$$Q_j^* = \begin{bmatrix} 0.36 & 0 & -0.36 & 0 & -0.96 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.36 & 0 & 0.36 & 0 & 0.96 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.2 \\ 0.48 & 0 & -0.48 & 0 & 0.72 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & -0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.36 & -0.36 & 0 & 0 & 0.96 \\ 0 & -0.36 & 0.36 & 0 & 0 & -0.96 \\ 0 & -0.48 & 0.48 & 0 & 0 & 0.72 \\ 0 & 0 & 0 & -0.6 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 & 0.6 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

for which

$$Q_j^{*T} Q_j^* = \begin{bmatrix} 1.4896 & 1 & 0.5104 & 0 & -0.3456 & 0 \\ 1 & 1.4896 & 0.5104 & 0 & 0 & 0.3456 \\ 0.5104 & 0.5104 & 1.9792 & 0 & 0.3456 & -0.3456 \\ 0 & 0 & 0 & 1.52 & 0 & 0 \\ -0.3456 & 0 & 0.3456 & 0 & 2.7616 & 0 \\ 0 & 0.3456 & -0.3456 & 0 & 0 & 2.7616 \end{bmatrix}$$

which is, it turns out, of full rank, and thus  $(Q_j^{*T} Q_j^*)^{-1}$  exists. Thus equation (4-13) will determine a unique set of values for the elements of

Equation (4-13) is not applicable to internal components directly. An analysis similar to that for the center of mass determination of the previous section results in the following equation for the inertia tensor elements of



internal components:

$$\begin{bmatrix} I_{j/C_j}'(1,1) \\ I_{j/C_j}'(2,2) \\ \vdots \\ I_{j/C_j}'(2,3) \end{bmatrix} = (Q_j^{*T} Q_j^*)^{-1} Q_j^{*T} \begin{bmatrix} \bar{b}_j^{I-II} \\ \bar{b}_j^{I-III} \\ s_j \end{bmatrix} - \sum_{\substack{i \text{ external} \\ \text{to } j}} \begin{bmatrix} \Delta^{I-II} (T_i I_{i/C_i}' T_i^T + P_i)(1,1) \\ \Delta^{I-II} (T_i I_{i/C_i}' T_i^T + P_i)(2,2) \\ \vdots \\ \Delta^{I-II} (T_i I_{i/C_i}' T_i^T + P_i)(2,3) \\ \hline \Delta^{I-III} (T_i I_{i/C_i}' T_i^T + P_i)(1,1) \\ \vdots \\ \Delta^{I-III} (T_i I_{i/C_i}' T_i^T + P_i)(2,3) \end{bmatrix} \quad (4-14)$$

The above equation is applicable only after the earlier presented methods have been used to determine  $\bar{R}_{C_j/J_j}$  and to determine  $\bar{R}_{C_i/J_i}$  and  $\Delta^{I-II}$  for  $i$  external to  $j$ . Equation (4-14) is actually generally applicable to any components (the summation is ignored for external components) and, of course, the central component.

The central component inertia tensor is obtainable from:

$$I_{n/C_n}' = T_n^T \left[ I_B - \sum_{i=1}^{n-1} (T_i I_{i/C_i}' T_i^T + P_i) - P_n \right] T_n \quad (4-15)$$

This last equation was obtained by simple manipulation of Equation (2-13).

The last two sections, then, have provided a means for finding the center of mass location and inertia tensor for each body component about component-fixed axes. A computer program capable of using this means (PROGRAM PART) is provided in Appendix B. Once the component properties are found, Equations (2-7) and (2-13) can be used to predict whole-body mass properties for any body position of a particular subject.

### Experimental Determination of Whole-Body Mass Properties

At the beginning of this chapter, it was seen that experimental data for whole-body mass properties for selected positions was needed in order to apply the method developed by this chapter for prediction of component mass properties. This section briefly describes how this whole-body mass property might be obtained.

While several researchers have experimentally determined centers of gravity for living subjects, only one determination of whole-body center of gravity and moments of inertia has been made (Santschi, DuBois, and Omoto). While whole-body products of inertia have not been experimentally determined, however, Fowler provides a means for carrying out such a determination.

Santschi, DuBois, and Omoto (Ref. 15) experimentally determined centers of gravity and moments of inertia about three axes for 66 living subjects in three positions. This was done by finding oscillation frequencies of a compound pendulum suspended alternately from two parallel axes. Mean values and standard deviations were determined for the subjects studied. Results were that useful predictions of moments of inertia for similar subjects could be made from height and weight alone.

The theoretical accuracy of the pendulum used was about 0.5 percent

for the center of gravity and between 2 and 5 percent for the moments of inertia. The length of the pendulum was long and the pendulum oscillations small ( $1^\circ$ ) to minimize the effect of shifting body fluids.

Fowler (Ref. 9) developed a theoretical method for experimentally determining spacecraft mass distribution while in space. By using a series of three thrusting maneuvers, the center of mass and inertia tensor of the craft can be determined, assuming the spacecraft mass is predetermined from a linear acceleration and that angular rates can be measured. This method is applicable to a human subject. If the experiment is done on earth, then known torques can replace the known thrusts. Though there is currently no device available with which to spin a human about three perpendicular axes (which Fowler's method requires), one can be developed.

In order to minimize the effect of redistribution of body fluids, angular rates should be kept as small as is accurately measurable.

If it is desired to include outer coverings in the values for mass properties, then the subject needs only to wear said coverings during the experimental tests (for either of the above described methods).

#### Contribution of Outer Garments or Suit

For some application, e.g. a maneuvering astronaut, the subject may wear an outer covering that moves with the body parts. The contribution of this outer covering to the system mass properties may not be negligible. The contribution of these outer garments or suit can be accounted for by using the equations already developed if we consider the outer covering of component  $j$ . No changes occur in the equations already given except that  $m_j$  and  $s_j$  will include both the body part and outer covering contribution. It is shown in Chapters 6 and 7 how  $m_j$  and  $s_j$  estimates

including the outer covering can be made. This method will be valid if the following two assumptions are made:

- (1) The components of the outer covering corresponding to the body components are themselves rigid bodies.
- (2) There is negligible relative motion between each component and its outer covering.

The basis for the solution for component mass properties is the formation of what will be called mass property difference equations. These are formed by subtracting the equations for whole-body mass properties (Equations (1-7) and (2-13)) for selected body positions. Using the center of mass difference equation and estimates for component masses, component centers of mass can be determined. Using the inertia tensor difference equation, estimates for component masses, and estimates for sums of component moments of inertia, component inertia tensors can be determined. It will be shown in Chapter 5 that estimates for sums of component moments of inertia provide an additional independent equation relating elements of component inertia tensors.

## CHAPTER 5

### Explanation of Infinity of Solutions for Component Inertia Tensors

In Chapter 4 it was claimed that the component inertia tensor could not be uniquely determined without the introduction of an additional equation relating the elements of the inertia tensor (the sum of component moments of inertia). The purpose of this chapter is to prove this claim.

Equation (4-8) can be expanded to include any number of positions as follows:

$$\underbrace{\begin{bmatrix} v_j^{I-II} \\ \hline v_j^{I-III} \\ \hline \vdots \\ \hline v_j^{I-N} \end{bmatrix}}_{\equiv Q_j} \begin{bmatrix} I_j'/c_j (1,1) \\ (2,2) \\ (3,3) \\ (1,2) \\ (1,3) \\ (2,3) \end{bmatrix} = \begin{bmatrix} \bar{b}_j^{I-II} \\ \hline \bar{b}_j^{I-III} \\ \hline \vdots \\ \hline \bar{b}_j^{I-N} \end{bmatrix} \quad (5-1)$$

where  $N$  is the number of body positions.

In Chapter 4 it was claimed that the inclusion of any number of body positions will not provide a  $Q_j^T Q_j$  of full rank, and thus (5-1) cannot be directly solved for the elements of  $I_j'/c_j$  for  $N$  positions selected. The purpose of this chapter is to prove this claim, which is equivalent to showing that there exists an infinity of solutions for the elements of  $I_j'/c_j$ .

in (5-1). Note that (5-1) is based solely on whole-body mass properties for selected positions and estimates for component masses.

The method of proof will be to show that for any positions I and II (only j different),  $V_j^{I-II}$  has the same linear dependence of columns, i.e. the sum of the first three columns is zero. Once this is shown, then it is easily seen that  $Q_j$  in (5-1) will have this linear dependence since each of the submatrices must.

Consider any orthogonal transformation T, which for this work is 3x3. By the definition of orthogonal,  $T^T = T^{-1}$ , and consequently  $TT^T = I$ . Writing this out in terms of elements and performing the matrix multiplication:

$$TT^T = \begin{bmatrix} T(1,1) & T(1,2) & T(1,3) \\ T(2,1) & T(2,2) & T(2,3) \\ T(3,1) & T(3,2) & T(3,3) \end{bmatrix} \begin{bmatrix} T(1,1) & T(2,1) & T(3,1) \\ T(1,2) & T(2,2) & T(3,2) \\ T(1,3) & T(2,3) & T(3,3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equivalently:

$$\begin{bmatrix} \begin{bmatrix} T^2(1,1) \\ +T^2(1,2) \\ +T^2(1,3) \end{bmatrix} & \begin{bmatrix} T(1,1)T(2,1) \\ +T(1,2)T(2,2) \\ +T(1,3)T(2,3) \end{bmatrix} & \begin{bmatrix} T(1,1)T(3,1) \\ +T(1,2)T(3,2) \\ +T(1,3)T(3,3) \end{bmatrix} \\ \begin{bmatrix} T(2,1)T(1,1) \\ +T(2,2)T(1,2) \\ +T(2,3)T(1,3) \end{bmatrix} & \begin{bmatrix} T^2(2,1) \\ +T^2(2,2) \\ +T^2(2,3) \end{bmatrix} & \begin{bmatrix} T(2,1)T(3,1) \\ +T(2,2)T(3,2) \\ +T(2,3)T(3,3) \end{bmatrix} \\ \begin{bmatrix} T(3,1)T(1,1) \\ +T(3,2)T(1,2) \\ +T(3,3)T(1,3) \end{bmatrix} & \begin{bmatrix} T(3,1)T(2,1) \\ +T(3,2)T(2,2) \\ +T(3,3)T(2,3) \end{bmatrix} & \begin{bmatrix} T^2(3,1) \\ +T^2(3,2) \\ +T^2(3,3) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and for any positions I and II:

$$T^I(1,1)^2 + T^I(1,2)^2 + T^I(1,3)^2 = T^{II}(1,1)^2 + T^{II}(1,2)^2 + T^{II}(1,3)^2 = 1$$

$$T^I(2,1)^2 + T^I(2,2)^2 + T^I(2,3)^2 = T^{II}(2,1)^2 + T^{II}(2,2)^2 + T^{II}(2,3)^2 = 1$$

$$T^I(3,1)^2 + T^I(3,2)^2 + T^I(3,3)^2 = T^{II}(3,1)^2 + T^{II}(3,2)^2 + T^{II}(3,3)^2 = 1$$

$$\begin{aligned} T^I(2,1)T^I(3,1) + T^I(2,2)T^I(3,2) + T^I(2,3)T^I(3,3) \\ = T^{II}(2,1)T^{II}(3,1) + T^{II}(2,2)T^{II}(3,2) + T^{II}(2,3)T^{II}(3,3) = 0 \end{aligned}$$

$$\begin{aligned} T^I(1,1)T^I(3,1) + T^I(1,2)T^I(3,2) + T^I(1,3)T^I(3,3) \\ = T^{II}(1,1)T^{II}(3,1) + T^{II}(1,2)T^{II}(3,2) + T^{II}(1,3)T^{II}(3,3) = 0 \end{aligned}$$

$$\begin{aligned} T^I(1,1)T^I(2,1) + T^I(1,2)T^I(2,2) + T^I(1,3)T^I(2,3) \\ = T^{II}(1,1)T^{II}(2,1) + T^{II}(1,2)T^{II}(2,2) + T^{II}(1,3)T^{II}(2,3) = 0 \end{aligned}$$

But manipulating the above equations, and then substituting the elements of  $V_j^{I-II}$  (referring to Equation (4-8):

$$V_j^{I-II}(1,1) + V_j^{I-II}(1,2) + V_j^{I-II}(1,3) = 0$$

$$V_j^{I-II}(2,1) + V_j^{I-II}(2,2) + V_j^{I-II}(2,3) = 0$$

$$V_j^{I-II}(3,1) + V_j^{I-II}(3,2) + V_j^{I-II}(3,3) = 0$$

$$V_j^{I-II}(4,1) + V_j^{I-II}(4,2) + V_j^{I-II}(4,3) = 0$$

$$V_j^{I-II}(5,1) + V_j^{I-II}(5,2) + V_j^{I-II}(5,3) = 0$$

$$V_j^{I-II}(6,1) + V_j^{I-II}(6,2) + V_j^{I-II}(6,3) = 0$$

It can be seen from these equations that the sum of the first three elements of  $V_j^{I-II}$  is zero. Thus  $V_j^{I-II}$  is of rank less than 6 for any positions I and II. The choices of positions I and II are arbitrary; therefore  $V_j^{I-II}$ ,  $V_j^{I-III}$ , ...  $V_j^{I-N}$  will all have this same linear depen-

ence of columns. Clearly, then,  $Q_j = \begin{bmatrix} v_j^{I-II} \\ \vdots \\ v_j^{I-N} \end{bmatrix}$  will be of rank less than

6 since the sum of its first three rows is zero, and thus it cannot have six linearly independent columns. Hence, for any number of positions, (5-1) will not yield a unique solution for the elements of  $I_{j/c_j}$ . In order to obtain a unique value, then, it is necessary to introduce an additional equation.

An approximation of the sum of the component moments of inertia can provide another independent equation, which together with a  $Q_j$  matrix formed from judicious selection of three body positions determine a unique value for the elements of  $I_{j/c_j}$  (as was shown in Example 3).



## CHAPTER 6

### Suggested Methods for Estimation of Component Masses

The determinations of the component centers of mass and inertia tensors depended on estimates for the masses of the components. The purpose of this chapter is to provide two possible methods for making this estimate. These are regression equations and water immersion (with an assumed density). Both methods depend on statistical data for cadavers. Inclusion of outer garments or suits is also considered.

#### Regression Equations

Estimates for component masses can be obtained from regression equations predicting component weights from total body weight.

J. T. Barter (Ref. 1) used the frozen cadaver data of earlier research-  
ers to devise a system of regression equations estimating the component  
weights. These equations are as follows, where all weights are in lbf.  
(Ref. 11, page 171):

Body Segment	Regression Equation	Standard Deviation of the Residuals
Head, neck and trunk	= 0.47 x Total body wt. + 5.4	(±2.9)
Total upper extremities	= 0.13 x Total body wt. - 1.4	(±1.0)
Both upper arms	= 0.08 x Total body wt. - 1.3	(±0.5)
Forearms plus hands <sup>a</sup>	= 0.06 x Total body wt. - 0.6	(±0.5)
Both forearms <sup>a</sup>	= 0.04 x Total body wt. - 0.2	(±0.5)
Both hands	= 0.01 x Total body wt. + 0.3	(±0.2)
Total lower extremities	= 0.31 x Total body wt. + 1.2	(±2.2)
Both upper legs	= 0.18 x Total body wt. + 1.5	(±1.6)
Both lower legs plus feet	= 0.13 x Total body wt. - 0.2	(±0.9)
Both lower legs	= 0.11 x Total body wt. - 0.9	(±0.7)
Both feet	= 0.02 x Total body wt. + 0.7	(±0.3)

<sup>a</sup> N = 11, all others N = 12.

The sum of the predicted weights of the above equations does not always equal the total body weight. To compensate for this, the difference is calculated, then distributed proportionally among the segments.

Another set of regression equations, also based on frozen cadaver data, was developed by Clauser and McConville (Ref. 4). They are as follows, where all weights are in lbf:

Head Weight	=	.028 Body Weight (W) + 6.354
Trunk Weight	=	.552 W - 6.417
Upper Arms Weight	=	.059 W + .862
Forearms Weight	=	.026 W + .85
Hands Weight	=	.009 W + .53
Upper Legs Weight	=	.239 W - 4.844
Lower Legs Weight	=	.067 W + 2.846
Feet Weight	=	.016 W + 1.826

As for Barter's equations, the predicted weights are corrected so that the difference between total body weight and the sum of predicted weights is distributed proportionally among the segments. Note that Clauser and McConville's equations provide an estimate for head weight not given by Barter's.

It is claimed that this second set of regression equations results in a substantial redistribution of weight. Which of the two sets of regression equations is more accurate is as yet an unanswered question.

### Water Immersion

By having the subject in question alternately immerse his body parts into water tanks, the volumes of components can be measured. Internal components' volumes can be found by immersing each internal component with the component(s) external to it, then subtracting the already determined volume(s) of the external component(s). The volume of the central component can be found by immersing the entire subject to find his/her whole-body volume, then subtracting the already determined volumes for the other body components.

Using statistical data for specific gravity of cadaver parts (Ref. 6, page 1955a), approximate values for component masses can be found from experimentally determined volumes by assuming that the subject's components' specific gravities are reasonably close to the cadaver data. However, this does not account for individual variations in body structure. Thus if one subject has larger than average bones then his component specific gravity may be expected to differ from the average data for specific gravity.

This method of water immersion may be qualified, in that component volume may be changed by water pressures encountered. This effect, if appreciable, can be minimized by immersing the components as much as possible

near the surface of the tank (more horizontal than vertical). One possible way to avoid such a deformation effect would be to use rigid casts of the subject's body parts for the immersion tests.

This water immersion technique, though more difficult and costly than using regression equations, has the advantage that it better accounts for individual differences in mass distribution.

#### Inclusion of Outer Garments or Suits in Component Masses

Each component mass can be considered to include the mass of the outer covering surrounding the component (see Chapter 4). The contribution to the component masses by the outer covering parts can be found by simply cutting the outer covering into appropriate sections and weighing. These values are then simply added to the corresponding component masses.

If destroying a garment or suit by cutting it into sections is economically undesirable or impossible, then the researcher must devise his own method for obtaining reasonable values for the masses of the outer covering components.

Tieber and Lindemuth provide a set of regression equations for estimating the component masses for a typical pressure suit (Ref. 17, page 28).

## CHAPTER 7

### Suggested Methods for Estimation of The Sums of Component Moments of Inertia

The determination of the component inertia tensors depended on estimates for the sums of component moments of inertia (see Chapters 4 and 5). The purpose of this chapter is to provide two possible methods for making this estimate. Methods for inclusion of outer garments or suit shall also be presented.

#### Hanavan Model

The first method is to obtain values for  $s_j$  ( $j = 1, \dots, n$ ) by using Hanavan's simply shaped components to obtain estimates for  $I_{j/c_j}'$  ( $j = 1, \dots, n$ ) (see Chapter 3) then finding  $s_j$  from its defining equation:

$$s_j \equiv \sum_{k=1}^3 I_{j/c_j}'(k, k) \quad (j = 1, \dots, n)$$

This is not equivalent to assuming simple geometric shapes for the body components, rather it is assuming that the sums of moments given by Hanavan's model will not differ significantly from the actual values for the sums.

Hanavan's estimates for  $I_{j/c_j}'$  ( $j = 1, \dots, n$ ) will, as was indicated in Chapter 3, depend upon measurements of the subject's dimensions and estimates for component masses. The researcher can use either of the methods presented in Chapter 6 to obtain component mass estimates.

### Casts of the Subject's Components

Molded casts of the particular subject's components made of a uniform density material provide the second method for estimating the  $s_j$  ( $j = 1, \dots, n$ ). If experimental tests are performed on these casts identical in nature to the method for whole-body tests discussed in Chapter 4, then estimates for  $I_j'/c_j$  ( $j = 1, \dots, n$ ), and thus for  $s_j$  ( $j = 1, \dots, n$ ) can be obtained.

It is not necessary that the uniform density material used to make the casts have the same specific gravity as the data for body components. This is true because, for example:

$$I_{xx} = \int_m (y^2 + z^2) dm = \int_v (y^2 + z^2) \rho dv = \rho \int_v (y^2 + z^2) dv$$

for the casts. Thus the ratio of specific gravities can be used as a scaling factor to obtain estimates for  $I_j'/c_j$  ( $j = 1, \dots, n$ ) for the subject.

It is worth noting that the orientation of the axes about which the spin tests are made is not critical as long as the three axes are mutually perpendicular. This is because the inertia tensor transforms by a similarity transformation. It can be easily proved that for an orthogonal similarity transformation, the sum of the main diagonal elements is invariant under any transformation.

### Inclusion of Outer Garments or Suits in $s_j$

There are two methods for the inclusion of outer garments or suits in the values for  $s_j$  ( $j = 1, \dots, n$ ).

The first is to use the Tieber-Lindemuth model for each suit part corresponding to a body component (see Chapter 3). This method can be used to find the inertia tensor for each suit part, which is then simply added to the inertia tensor for the corresponding body component (obtained either

by using Hanavan's model or cast experimental data).

The second method involves actually cutting up the garment or suit into segments corresponding to the body parts and performing experimental tests to obtain inertia tensors. This can be done separately and adding in or actually surrounding the corresponding cast to obtain  $I_j'/C_j$  ( $j = 1, \dots, n$ ) for the combined body component and suit segment. For flexible outer coverings, such test data may be of questionable validity. For a pressure suit, it may be possible to inflate each component to sufficient pressure to render it sufficiently rigid to solve the problem of flexibility. This would involve somehow sealing off the ends of the section of pressure suit, however.

### Conclusion

This chapter has presented two possible methods for obtaining reasonable estimates for the sums of component moments of inertia (with or without outer coverings or suits). Both methods presented depended on statistical data for cadavers (Hanavan's model on the component mass estimates; the component cast method on the specific gravity data).

Hanavan's model is clearly the easiest and least expensive, as the only experimental data needed is the total body mass and the various anthropometric data. But the component cast method, while more costly and more difficult to carry out, offers the advantage of actually taking the possible variations from statistical average of component shapes into account.

## CHAPTER 8

### Limitations

There are four types of limitations on the method for predicting mass properties presented in Chapter 4. These are errors introduced by invalidity in assumptions, errors in whole-body experimental data, accumulated errors in the mass property predictions for the central component, and joint constraints preventing exact application of the method. The remainder of this chapter discusses these and presents recommendations for further study where appropriate. Finally, suggestions for future sensitivity analyses are made.

#### Invalidity of Assumptions

There is some question as to the validity of each of the assumptions made in the development of the present study's method for determination of component mass properties.

As was indicated in Chapter 1, the rigid body assumption is of marginal validity for the torso because of the flexibility of the backbone. Hanavan's model partially models this by dividing the torso into two halves. Theoretically, nothing precludes such a division of the central component for this study's method. It remains for future researchers to determine the validity of such a representation of the central component. For a pressure-suited astronaut, mobility is restricted sufficiently that the torso may prove to be adequately modeled by only one component.

The hands and feet are also flexible, thus also compromising the rigid body assumption. But these body parts are of sufficiently small mass that



the variation in contributions to the mass properties due to flexibility may be negligible. This is even more likely for a pressure-suited astronaut with reduced limb mobility. This also remains for future researchers to determine.

The assumption that there exist valid methods for estimating component masses depends on there being little significant difference between component masses of a live subject and a frozen cadaver. This is the case only if component properties are approximately the same before and after death and if individual variations are not too great. Some indication of the validity of the estimation methods would be given by comparing results after application of both methods. Also, because the total number in the sample of cadaver data is, to date, relatively small, there is a need for more such data to increase the validity of the regression equations and specific gravity data. Further, as was indicated in Chapter 6, the effect of individual variations in body structure and of water pressure on component volumes in water immersion experiments must be considered and investigated.

Next, the validity of methods of obtaining sums of component moments of inertia is considered. Both methods presented (the Hanavan model and cast data) require estimated values for component masses which may themselves be in error. Also both methods use uniform density components, which actual human components are not. That there is negligible difference is another subject for future study.

Finally, the assumption that the components of the outer covering are inflexible and that there is negligible relative motion between each of the body components and the corresponding outer covering component introduce negligible error should also be experimentally investigated.

### Errors in Whole-Body Experimental Data

Errors in whole-body experimental data arise from shifting body fluids, experimental measurement errors (in oscillation frequencies or angular rates and torques), movement by the subject, and the environment of the experiments.

The error due to shifting body fluids can be minimized, as was indicated in Chapter 4, by small oscillation amplitudes and long pendulum length (for the complex pendulum) and by small angular rates (for Fowler's method). Future work would involve improvement of the complex pendulum accuracy and development of a device capable of physically realizing Fowler's method.

Inadvertent movements by the subject can be minimized by careful design of the apparatus used. Some movement is unavoidable, however, specifically internal movement of blood and expansion and contraction of the lungs. The influence of these must also be investigated in the future.

The environment of the whole-body experiments may also be of some importance. Probably most significant in this respect is the effect of gravity. If the experiments are carried out in the presence of appreciable gravity (they must be for the complex pendulum), then the gravitational force will tend to settle the body fluids toward the lower portion of the body. Thus an astronaut under weightless conditions may have an appreciably different mass distribution than on earth. This effect may be negligible, but should be investigated. One avenue of study would be to perform the experiments with the subject both upside down and right side up and compare the resulting data for mass properties.

### Accumulated Errors in Central Component

As was indicated in Chapter 4, the central component mass properties

can be determined only after the mass properties of all other components have been found. Thus any errors made in the determination of the component mass properties will accumulate in finding the mass properties for the central component. There is, of course, the possibility that errors in the component mass properties will cancel out, but this cannot be assumed.

### Joint Constraints

The bending of certain body joints is constrained so as to preclude the orientation of the external components in three positions not in one plane (this was suspected in Chapter 4 as necessary to uniquely determine the component mass properties). The elbows and knees (hinge joints) are joints of this type.

For the joints immediately external to these joints, then, a researcher can only obtain approximate values by slightly rotating the components internal to the hinge joint for one of the selected positions so as to allow the external component to get out of the plane of the other positions. Because both the thighs and upper arms appear fairly symmetric, this seems a reasonable solution to the problem of joint constraints. Experimental data on cadaver thighs and upper arms can serve as a more quantitative measure of this symmetry.

### Suggested Sensitivity Analyses

Sensitivity analyses on the effect of the possible sources of error presented in this chapter should be carried out in the future so as to determine accuracy requirements for reasonable predicted results. These would consist of alternately determining the relationship between input errors in component masses, sums of component moments, and whole-body data

and the resulting differences in predictions output. Of secondary importance, though, is the differences in predicted mass propertires; more important are the differences in predicted dynamics of the system of interest.

## CHAPTER 9

### Conclusion

In this study a method has been presented for predicting the component mass properties for a human about component-fixed axes, with or without outer coverings. This prediction depended on a component rigid body assumption, estimates of component masses and sums of component moments of inertia, and data for whole-body mass properties for selected, known positions. If an outer covering was to be included, then the prediction additionally depended on the assumptions that negligible error was introduced by considering the covering components rigid and relative movement between body and covering components negligible.

The researcher is free to choose the number and definitions of the body components. Thus if it turns out that significant error does not result, the hands might be considered part of the forearm, the feet part of the calves, or the head and neck part of the trunk. It remains for future researchers to investigate the validity of different component definitions.

Limitations of the results including several possible sources of error were discussed in the previous chapter. Some of these error sources are not easily isolated nor the degree of error easily ascertained. However, the best indication of the validity of the method presented in this study is not the accuracy of the predicted mass properties, but the degree of agreement between predicted and actual dynamics of the system of interest.

In summary of the method presented in this study, a flow chart is provided in Figure 4 indicating the sequence of steps taken in actually

carrying out a prediction of the mass properties for a component. Dotted lines indicated optional steps. Once component mass properties have been determined, whole-body mass properties can be predicted for any position by using Equations (2-7) and (2-13).

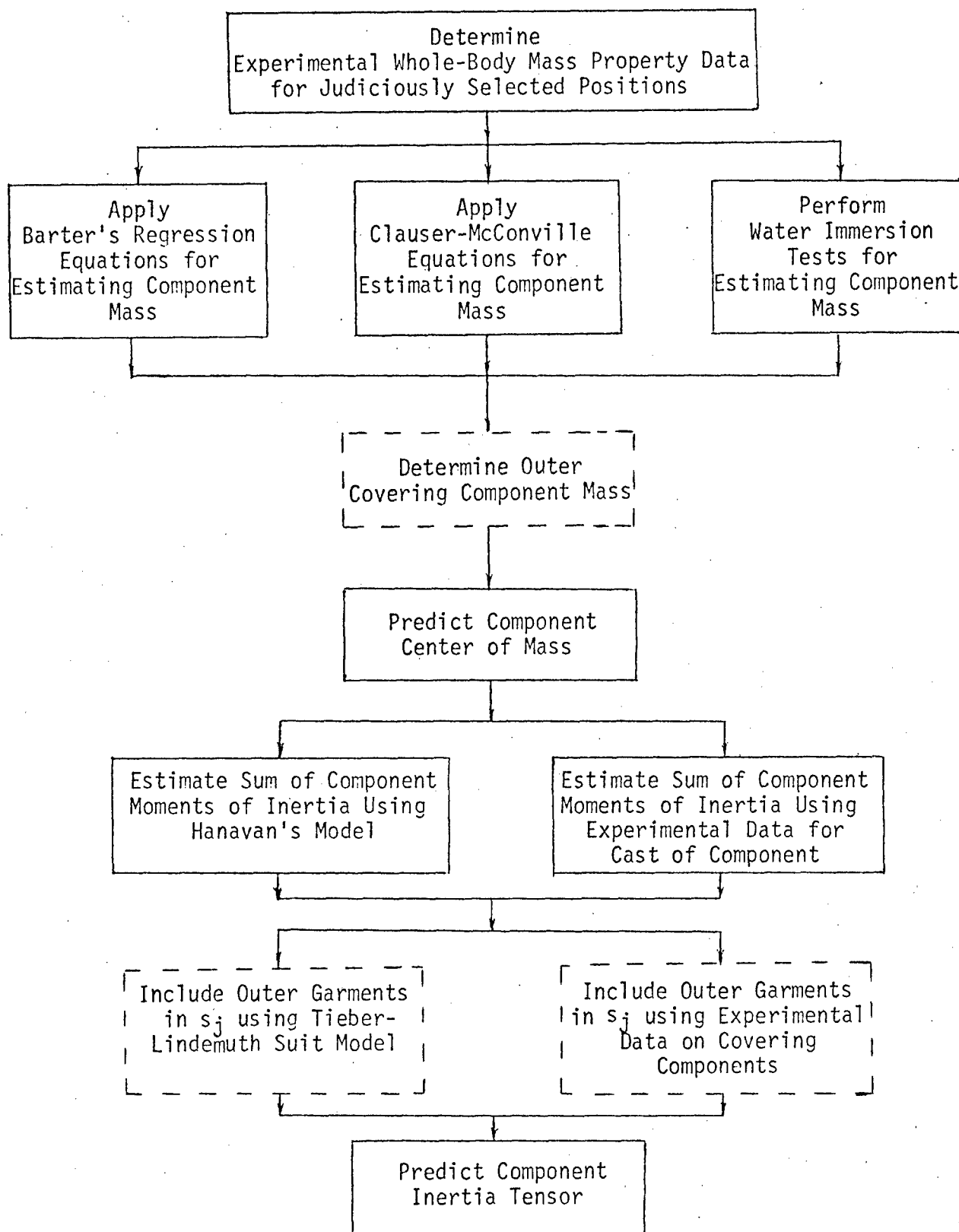


Figure 4. Flow Chart of the Determination of Component Mass Properties

# APPENDIX A

## PROGRAM WHOLE (FORTRAN IV)

The purpose of PROGRAM WHOLE is to calculate whole-body mass properties from component mass properties about component-fixed axes. An explanation of the symbols used in the program follows:

<u>Computer Symbol</u>	<u>Study Symbol and/or Explanation</u>
A	The Point A.
FCTR	Conversion Factor from Degrees to Radians.
G	The Angle $\gamma$
IBA(3,3)	The Matrix $I_{B/A}$ .
ICA(3,3)	The Matrix $I_{i/A}$ .
ICC(3,3)	The Matrix $I_{i/C_i}$ .
ICCP(3,3)	The Matrix $I_i' / C_i$ .
MB	The Scalar $m_B$ .
MI	The Scalar $m_i$ .
P(3,3)	The Matrix $P_i$ .
PI	$\pi$
PRMAT	A Matrix-Printing Subroutine.
RBA(3)	The Vector $\bar{R}_{B/A}$ .
RCA(3)	The Vector $\bar{R}_{C_i/A}$ .
RCJ(3)	The Vector $\bar{R}_{C_i/J_i}$ .
RCJP(3)	The Vector $\bar{R}_{C_i}' / J_i$ .
RJA(3)	The Vector $\bar{R}_{J_i}$ .
T(3,3)	The Matrix $T_i$ .
TICCP(3,3)	The Matrix $T_i$ Times $I_i' / C_i$ .



U The Angle  $\mu$ .

X The Angle  $\chi$ .

The following are external subroutines (listing is not provided for brevity):

<u>External Subroutine</u>	<u>Purpose</u>
MSAB	Matrix Multiplication (AB).
MSABT	Matrix Multiplication ( $AB^T$ ).
MATADD	Matrix Addition.
MSNULL	Sets all the Elements of a Matrix to zero.

A listing of PROGRAM WHOLE follows.

## PROGRAM WHOLE (INPUT, OUTPUT)

```

*****
*
*   THIS PROGRAM CALCULATES THE FOLLOWING
*
*   1) COMPONENT AND WHOLE-BODY CENTER OF MASS
*       LOCATIONS RELATIVE TO ARBITRARY POINT A,
*
*   2) COMPONENT AND WHOLE-BODY INERTIA MATRICES
*       ABOUT THE AXES SYSTEM AT A.
*
*   UNITS      ---
*
*   A LOCATION  ---
*
*   BODY POSITION ---
*
*****

```

```

      DIMENSION RCJP(3), RCJ(3), PCA(3), RJA(3), RBA(3),
2 T(3,3), P(3,3), TICCP(3,3)
      REAL MI, MB, ICCP(3,3), ICC(3,3), ICA(3,3), IBA(3,3)
      PI = 3.14159265358979
      FCTR = PI/180.
      READ 200, MB, N
      CALL MSNULL (PRA, 3, 1, 3)
      CALL MSNULL (TRA, 3, 3, 3)
      PRINT 102
102  FORMAT (1H1)
      DO 1 I = 1, N
      READ 203, MI
      READ 201, X, G, U
      PRINT 100, X, G, U
      X = X*FCTR      $      G = G*FCTR      $      U = U*FCTR
100  FORMAT (5X, 3F10.6)
      SINX = SIN(X)      $      COSX = COS(X)
      SING = SIN(G)      $      COSG = COS(G)
      SINU = SIN(U)      $      COSU = COS(U)
      T(1,1) = COSX*COSG
      T(1,2) = COSX*SING*SINU - SINX*COSU
      T(1,3) = COSX*SING*COSU + SINX*SINU
      T(2,1) = SINX*COSG
      T(2,2) = SINX*SING*SINU + COSX*COSU
      T(2,3) = SINX*SING*COSU - COSX*SINU
      T(3,1) = -SING
      T(3,2) = COSG*SINU
      T(3,3) = COSG*COSU
      READ 202, RCJP
      READ 202, RJA

```

```

      CALL MSAB (T, 3, 3, RCIP, 3, 1, RCJ, 3, 3, 3)
      CALL MATADD (RCJ, RJA, RCA, 3, 1, 3, 3, 3)
      DO 2 K = 1, 3
        RRA(K) = RRA(K) + RCA(K)*(MI/MB)
2     CONTINUE
      DO 3 K = 1, 3
        READ 202, (ICCP(K,L), L=1,3)
3     CONTINUE
      CALL MSAB (T, 3, 3, ICCP, 3, 3, TICCP, 3, 3, 3)
      CALL MSABT (TICCP, 3, 3, T, 3, 3, ICC, 3, 3, 3)
      P(1,1) = MI * (RCA(2)*RCA(2) + RCA(3)*RCA(3))
      P(2,2) = MI * (RCA(1)*RCA(1) + RCA(3)*RCA(3))
      P(3,3) = MI * (RCA(1)*RCA(1) + RCA(2)*RCA(2))
      P(1,2) = MI * (RCA(1)*RCA(2))
      P(1,3) = MI * (RCA(1)*RCA(3))
      P(2,3) = MI * (RCA(2)*RCA(3))
      P(2,1) = P(1,2)
      P(3,1) = P(1,3)
      P(3,2) = P(2,3)
      CALL MATADD (ICC, P, ICA, 3, 3, 3, 3, 3)
      CALL MATADD (IRA, ICA, IRA, 3, 3, 3, 3, 3)
1     CONTINUE
      CALL PRMAT (RRA, 3, 1)
      CALL PRMAT (IRA, 3, 3)
200  FORMAT (F10.6, 110)
201  FORMAT (3F10.2)
202  FORMAT (3F20.6)
203  FORMAT (F10.6)
      END

```

```

      SUBROUTINE PRMAT (ATRIX, K,L)
      *****
      * THIS SUBROUTINE PRINTS ANY K BY L MATRIX. *
      *****
      DIMENSION ATRIX (K,L)
      PRINT 101
      DO 1 M = 1,K
        PRINT 100, (ATRIX(M,N), N = 1,L)
1     CONTINUE
100  FORMAT (1X, 6F20.12)
101  FORMAT (1/, 1X, 71(14*))
      RETURN      $      END

```

## APPENDIX B

### PROGRAM PART (FORTRAN IV)

The purpose of PROGRAM PART is to calculate the mass properties for any body component (except the central component) from whole-body mass property data for three selected positions, an estimate of the component mass, and an estimate of the sum of the component moments of inertia. An explanation of important symbols different from or in addition to those symbols of PROGRAM WHOLE (Appendix A) follows:

<u>Computer Symbol</u>	<u>Study Symbol and/or explanation</u>
A(6,3)	The Matrix $A_j$
ATA(3,3)	The Matrix $A_j^T A_j$
ATAAT(3,6)	The Matrix $\left[A_j^T A_j\right]^{-1} A_j^T$
BBI(6)	A Dummy Vector Needed by Subroutine MATINV.
BBR(3)	A Dummy Vector Needed by Subroutine MATINV.
BI(13)	The Vector $\begin{bmatrix} \overline{b}_j^{I-II} \\ \hline \overline{b}_j^{I-III} \\ \hline s_j \end{bmatrix}$
BR(6)	The Vector $\begin{bmatrix} \Delta^{I-II}(\overline{R}_B) \\ \hline \Delta^{I-III}(\overline{R}_B) \end{bmatrix}$
DETER	The Determinant of the Matrix Inverted by MATINV.

The Prefix E

$E(6)$  ( $=E1+E2$ )

$E0(6,3)$

$E1(6)$  ( $=(E0)(ERCJP)$ )

$E2(6)$

$EI(13)$

MC

The Suffix N

NE

$Q(13,6)$

$QTQ(6,6)$

$QTQQT$

SC

$V12(6,6)$

$V13(6,6)$

The Computer Symbol Prefixed is for a Component External to the Component Whose Mass Properties are Sought.

A Vector Used for Notational Brevity.

A Matrix Used for Notational Brevity.

A Vector Used for Notational Brevity.

A Vector Used for Notational Brevity.

A Vector Used for Notational Brevity.

$m_j$

The Symbol Suffixed if for Body Position N.

The Number of Components External to the Component Whose Mass Properties Being Sought.

$Q_j^*$

$Q_j^{*T} Q_j^*$  OR  $(Q_j^{*T} Q_j^*)^{-1}$

$(Q_j^{*T} Q_j^*)^{-1} Q_j^T$

$s_j$

$v_j^{I-II}$

$v_j^{I-III}$

The following are additional external subroutines (listing is not provided for brevity):

External SubroutinePurpose

MATINV

Matrix Inversion

MSATB

Matrix Multiplication ( $A^T B$ )

VSUBC

Vector Subtraction

## PROGRAM PART (INPUT, OUTPUT)

```

*****
*
*   THIS PROGRAM DETERMINES THE MASS PROPERTIES FOR
*   ANY COMPONENT OF THE BODY EXCEPT FOR THE CENTRAL
*   COMPONENT (TYPICALLY THE TORSO).
*
*****

```

```

      DIMENSION A(6,3), ATA(3,3), ATAAT(3,6), BRI(6), BRR(3),
2  RI(13), RR(6), F(6), F0(6,3), E1(6), E2(6), EI(13),
3  EICCP(3,3), EP(3,3,3), ERCJP(3), ERJA(3,3), ET(3,3,3),
4  ETITN(3,3), ETITT(3,3,3), FTN(3,3), IRA(3,3,3),
5  ICCP(3,3), INDER(3,2), INDEFI(6,2), IPIVR(6), IPIVI(6),
6  P(3,3,3), Q(13,6), QTQ(6,6), QTQQT(6,13), R(3),
7  RRA(3,3), RCJP(3), RJA(3), SUMR(6), SUMI(13), T(3,3,3),
8  V12(6,6), V13(6,6), VCTR(6)
      REAL MR, MC, MF, IRA, ICCP
      PI = 3.14159265358979
      FCTR = PI/180.
      PRINT 204
      READ 200, MR, MF
      READ 203, MC
      DO 1 N = 1, 3
      READ 201, X, G, U
      PRINT 201, X, G, U
      X = X*FCTR      $      G = G*FCTR      $      U = U*FCTR
1  CALL FORMT (T, N, X, G, U)
      DO 2 K = 1, 3
      DO 2 L = 1, 3
      A(K,L) = T(1,K,L) - T(2,K,L)
2  A(K+3,L) = T(1,K,L) - T(3,K,L)
      CALL MSATR (A, 6, 3, A, 6, 3, ATA, 6, 6, 3)
      CALL MATINV (ATA,3,BRR,0,DETER,IPIVR,INDER,3,ISCAR)
      CALL MSABT (ATA, 3, 3, A, 6, 3, ATAAT, 3, 6, 3)
      DO 3 N = 1, 3
      READ 202, R
      DO 3 K = 1, 3
3  RRA(N,K) = R(K)
      DO 5 K = 1, 3
      RR(K) = (MR/MC) * (RRA(1,K) - RRA(2,K))
5  RR(K+3) = (MR/MC) * (RRA(1,K) - RRA(3,K))
      IF (NE.EQ.0) GO TO 13
      DO 12 I = 1, NF
      READ 203, MF
      DO 6 K = 1,6
6  SUMR(K) = 0.0
      DO 7 N = 1,3
      READ 201, EX, EG, FI)

```

```

EX = EX*FCTR      $      EG = EG*FCTR      $      EU = EU*FCTR
7 CALL FORMT (ET, N, EX, EG, EU)
DO 8 K = 1, 3
DO 8 L = 1, 3
E0(K,L) = (ME/MC) * (ET(1,K,L) - ET(2,K,L))
8 E0(K+3,L) = (ME/MC) * (ET(1,K,L) - ET(3,K,L))
READ 202, ERCJP
CALL MSAB (E0, 6, 3, ERCJP, 3, 1, E1, 6, 3, 3)
DO 9 N = 1, 3
READ 202, R
DO 9 K = 1, 3
9 ERJA(N,K) = R(K)
DO 11 K = 1, 3
E2(K) = (ME/MC) * (ERJA(1,K) - ERJA(2,K))
11 E2(K+3) = (ME/MC) * (ERJA(1,K) - ERJA(3,K))
CALL VADDC (E1, E2, F, 6)
CALL VADDC (SUMR, F, SUMR, 6)
DO 21 K = 1, 13
21 SUMI(K) = 0.0
DO 18 K = 1, 3
READ 202, R
DO 18 L = 1, 3
18 EICCP(K,L) = R(L)
DO 20 N = 1, 3
DO 19 K = 1, 3
DO 19 L = 1, 3
19 ETN(K,L) = ET(N,K,L)
CALL MSAB (ETN, 3, 3, EICCP, 3, 3, ETI, 3, 3, 3)
CALL MSABT (ETI, 3, 3, ETN, 3, 3, ETITN, 3, 3, 3)
DO 23 K = 1, 3
DO 23 L = 1, 3
23 ETITT(N,K,L) = ETITN(K,L)
20 CALL PAXTH (EP, N, ET, ME, ERCJP, ERJA)
DO 22 K = 1, 3
KP = 4-K
CALL PQR(KP, KQ, KR)
EI(K) = ETITT(1,K,K) - ETITT(2,K,K) + EP(1,K,K) - EP(2,K,K)
EI(K+6) = ETITT(1,K,K) - ETITT(2,K,K) + EP(1,K,K) - EP(3,K,K)
22 EI(K+9) = ETITT(1,KQ,KR) - ETITT(3,KQ,KR) + EP(1,KQ,KR) - EP(3,KQ,KR)
EI(13) = 0.0
CALL VADDC (SUMI, EI, SUMI, 13)
12 CONTINUE
CALL VSUBC (BR, SUMR, BR, 6)
CALL VSUBC (BI, SUMI, BI, 13)
13 CONTINUE
CALL MSAB (ATAAT, 3, 6, BR, 6, 1, RCJP, 3, 6, 3)
CALL PRMAT (RCJP, 3, 1)
CALL FORMV (V12, T, 1, 2)
CALL FORMV (V13, T, 1, 3)
CALL FORMQ (Q, V12, V13)

```



```

CALL PRMAT (Q, 13, 6)
CALL MSATR (Q, 13, 6, Q, 13, 6, QTQ, 13, 13, 6)
CALL PRMAT (QTQ, 6, 6)
CALL MATINV (QTQ, 6, BRI, 0, DETER, IPIVI, INDEI, 6, ISCAI)
CALL PRMAT (QTQ, 6, 6)
CALL MSABT (QTQ, 6, 6, Q, 13, 6, QTQQT, 6, 13, 6)
READ 202, RJA
CALL PRMAT (RJA, 3, 1)
PRINT 203, MC
DO 14 N = 1, 3
CALL SIXTH (P, N, T, MC, RCJP, RJA)
DO 14 K = 1, 3
READ 202, R
DO 14 L = 1, 3
14 IRA (N,K,L) = P(L)
DO 15 K = 1, 3
RI(K) = IRA(1,K,K) - IRA(2,K,K) - P(1,K,K) + P(2,K,K)
RI(K+6) = IRA(1,K,K) - IRA(3,K,K) - P(1,K,K) + P(3,K,K)
KP = 4-K
CALL POR (KP, KO, KR)
BI(K+3) = IRA(1,KO,KR) - IRA(2,KO,KR) - P(1,KO,KR) + P(2,KO,KR)
15 BI(K+9) = IRA(1,KO,KR) - IRA(3,KO,KR) - P(1,KO,KR) + P(3,KO,KR)
READ 203, SC
BI(13) = SC
CALL PRMAT (BI, 13, 1)
CALL MSAB (QTQQT, 6, 13, BI, 13, 1, VCTRI, 6, 13, 6)
DO 17 K = 1, 3
17 ICCP(K,K) = VCTRI(K)
ICCP(1,2) = VCTRI(4)
ICCP(1,3) = VCTRI(5)
ICCP(2,3) = VCTRI(6)
CALL PRMAT (ICCP, 3, 3)
200 FORMAT (F10.6, I10)
201 FORMAT (3F10.2)
202 FORMAT (3F20.6)
203 FORMAT (F10.6)
204 FORMAT (1H1)
END

```

```

      SUBROUTINE FORMT(T, N, X, G, U)
      *****
      * THIS SUBROUTINE FORMS THE MATRIX T.
      *****
      DIMENSION T(3,3,3)
      SINX = SIN(X)      $      COSX = COS(X)
      SING = SIN(G)      $      COSG = COS(G)
      SINU = SIN(U)      $      COSU = COS(U)
      T(N,1,1)=COSX*COSG
      T(N,1,2)=COSX*SING*SINU - SINX*COSU
      T(N,1,3)=COSX*SING*COSU + SINX*SINU
      T(N,2,1)=SINX*COSG
      T(N,2,2)=SINX*SING*SINU + COSX*COSU
      T(N,2,3)=SINX*SING*COSU - COSX*SINU
      T(N,3,1)=-SING
      T(N,3,2)=COSG*SINU
      T(N,3,3)=COSG*COSU
      RETURN      $      END

```

```

      SUBROUTINE FORMQ(Q, V12, V13)
      *****
      * THIS SUBROUTINE FORMS THE MATRIX Q.
      *****
      DIMENSION Q(13,6), V12(6,6), V13(6,6)
      DO 1 K = 1, 6
      DO 1 L = 1, 6
      Q(K,L) = V12(K,L)
      1 Q(K+6,L)=V13(K,L)
      Q(13,1) = 1.0      $      Q(13,2) = 1.0      $      Q(13,3) = 1.0
      Q(13,4) = 0.0      $      Q(13,5) = 0.0      $      Q(13,6) = 0.0
      RETURN      $      END

```

```

      SUBROUTINE FORMV(V, T, M, N)
      *****
      * THIS SUBROUTINE FORMS THE MATRIX V.
      *****
      DIMENSION V(6,6), T(3,3,3)
      DO 1 K = 1, 3
      DO 1 L = 1, 3
      1 V(K,L) = T(M,K,1)*T(M,K,L) - T(N,K,L)*T(N,K,L)
      DO 2 L = 4, 6
      LP = 7-L
      CALL POR(LP, 10, LR)
      DO 2 K = 1, 3
      2 V(K,L)=2.0*(T(M,K,10)*T(M,K,LR)-T(N,K,10)*T(N,K,LP))
      DO 3 K = 4, 6

```

```

      KP = K-6
      CALL PQR (KP, KQ, KR)
      DO 3 L = 1, 3
3     V(K,L) = T(M,KQ,L)*T(M,KR,L) - T(N,KQ,L)*T(N,KR,L)
      DO 4 K = 4, 6
      KP = K-3
      CALL PQR (KP, KQ, KR)
      DO 4 L = 4, 6
      LP = 7-L
      CALL PQR (LP, LQ, LR)
4     V(K,L) = T(M,KQ,LR)*T(M,KR,LQ) + T(M,KQ,LQ)*T(M,KR,LR)
      2     -T(N,KQ,LR)*T(N,KR,LQ) - T(N,KQ,LQ)*T(N,KR,LR)
      RETURN      $      END

```

```

      SUBROUTINE PAXTH (P, N, T, MC, RCJP, RJA)
      *****
      * THIS SUBROUTINE APPLIES THE PARALLEL AXIS THEOREM. *
      *****
      DIMENSION P(3,3,3), PN(3,3), RCA(3), RCJ(3), RCJP(3),
      2 RJA(3), T(3,3,3), TN(3,3)
      REAL MC
      PRINT 101, MC
      CALL PRMAT (RCJP, 3, 1)
      CALL PRMAT (RJA, 3, 1)
      DO 1 K = 1, 3
      DO 1 L = 1, 3
1     TN(K,L) = T(N,K,L)
      CALL PRMAT (TN, 3, 3)
      CALL MSAB (TN, 3, 3, RCJP, 3, 1, RCJ, 3, 3, 3)
      CALL PRMAT (RCJ, 3, 1)
      CALL VADDC (RCJ, RJA, RCA, 3)
      CALL PRMAT (RCA, 3, 1)
      DO 2 IP = 1, 3
      CALL PQR (IP, IQ, IR)
      PRINT 100, IP, IQ, IR
100  FORMAT (10X, 3I6)
      P(N,IP,IP) = MC* (RCA(IQ)*RCA(IQ) + RCA(IR)*RCA(IR))
      P(N,IQ,IR) = MC* (RCA(IQ)*RCA(IR))
      P(N,IR,IQ) = P(N,IQ,IR)
      PRINT 01, P(N,IP,IP), P(N,IQ,IR)
101  FORMAT (10X, 2F20.12)
      2 CONTINUE
      DO 3 K = 1, 3
      DO 3 L = 1, 3
3     PN(K,L) = P(N,K,L)
      CALL PRMAT (PN, 3, 3)
      RETURN      $      END

```

SUBROUTINE PQR (IP, IQ, IR)

\*\*\*\*\*

\* THIS SUBROUTINE DETERMINES THE INTEGERS Q AND R FROM P.\*

\*\*\*\*\*

IF (IP.EQ.1) GO TO 1

IF (IP.EQ.2) GO TO 2

IF (IP.EQ.3) GO TO 3

1 IQ=2       \$       IR = 2       \$       GO TO 4

2 IQ = 1       \$       IR = 3       \$       GO TO 4

3 IQ = 1       \$       IR = 2

4 CONTINUE

RETURN       \$       END

SUBROUTINE PRMAT (ATRIX, K,L)

\*\*\*\*\*

\* THIS SUBROUTINE PRINTS ANY K BY L MATRIX.       \*

\*\*\*\*\*

DIMENSION ATRIX (K,L)

PRINT 101

DO 1 M = 1,K

1 PRINT 100, (ATRIX(M,N), N = 1,L)

100 FORMAT (1X, 6F20.12)

101 FORMAT (1/, 1X, 71(14\*))

RETURN       \$       END

## BIBLIOGRAPHY

1. Barter, J. T., Estimation of the Mass of Body Segments, WADC Technical Report 57-260, Wright-Patterson Air Force Base, Ohio: Wright Air Development Center, April, 1957.
2. Braune, W., and O. Fisher, "Bestimmung der Trägheitsmomente des menschlichen Körpers und seiner Glieder" ("Determination of Moment of Inertia of the Human Body and Its Limbs"), Abh. d. math.-phys. Cl. d. k. Sachs. Gesellsch. d. Wissen., 18:409 (Leipzig, 1892).
3. Braune, W., and O. Fischer, "Über den Schwerpunkt des menschlichen Körpers mit Rücksicht auf die Ausrüstung des deutschen Infanteristen" ("Center of Gravity of the Human Body as Related to the Equipment of the German Infantry"), Abh. d. math.-phys. Cl. d. k. Sachs. Gesellsch. d. Wissen., 15:561 (Leipzig, 1889).
4. Clauser, C., and J. McConville, Unpublished data from study to determine selected characteristics of segments of the human body, Wright-Patterson Air Force Base, Ohio: Aerospace Medical Research Laboratories, 1964-1965.
5. Damon, Albert, Howard W. Stoudt, and Ross A. McFarland, The Human Body in Equipment Design, Harvard University Press, Cambridge, Massachusetts, 1966.
6. Dempster, W. T., The Anthropology of Body Action, WADD Technical Report 60-18, Wright-Patterson Air Force Base, Ohio: Wright Air Development Division, January, 1960.
7. DuBois, J., et al., Moments of Inertia and Centers of Gravity of the Living Human Body Encumbered by a Full Pressure Suit, Wright-Patterson Air Force Base, Ohio: 6570th Aerospace Medical Research Laboratories, 1964.
8. Fischer, O., Theoretische Grundlagen fuer eine Mechanik der Lebenden Koerper (Theoretical Fundamentals for a Mechanics of Living Bodies with Special Application to Man as Well as to Some Processes of Motion in Machines), Berlin: B. G. Teubner, 1906.
9. Fowler, Wallace T., "In-Flight Determination of Spacecraft Mass Distribution for Lifting Body Reentry," The University of Texas at Austin, unpublished, June, 1968.
10. Halfman, Robert L., Dynamics, Particles, Rigid Bodies, and Systems, Volume I, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962.
11. Hanavan, Ernest P., A Mathematical Model of the Human Body, AMRL Technical Report 64-102, Wright-Patterson Air Force Base, Ohio: Aerospace Medical Research Laboratory, October, 1964.

12. King, B. G., et al., The Center of Mass of Man, NAVTRADEVCECEN Technical Report No. 85, Port Washington, New York: Naval Training Device Center, November, 1959.
13. Miele, Angelo, Flight Mechanics, Volume I, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962.
14. Noble, Ben, Applied Linear Algebra, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1969.
15. Santschi, W.R., et al., Moments of Inertia and Centers of Gravity of the Living Human Body, AMRL Technical Documentary Report 63-36, Wright-Patterson Air Force Base, Ohio: 6570th Aerospace Medical Research Laboratories, May, 1963.
16. Swearingen, J.J., Determination of the Centers of Gravity of Man, C.A.A. Project No. 53-203, Norman, Oklahoma: Civil Aviation Medical Research Laboratory, Aeronautical Center, Civil Aeronautics Administration, May, 1953.
17. Tieber, Julius A., and Robert W. Lindemuth, An Analysis of the Inertial Properties and Performance of the Astronaut Maneuvering System, AMRL Technical Report 65-4, Wright-Patterson Air Force Base, Ohio: Aerospace Medical Research Laboratory, August, 1965.

## VITA

Robert Daniel Watkins was born in Washington, D.C., on August 23, 1951, the son of the present Major General James Howard and Mary Kathleen Watkins. After completing his work at Washington-Lee High School, Arlington, Virginia, in 1969, he entered The University of Texas at Austin. In May, 1973, he married Bonnie Higgins, of Houston, Texas. He received the degree of Bachelor of Science in Engineering Science from The University of Texas in December, 1973, and was commissioned a second lieutenant in the United States Air Force. In May, 1974, he officially entered The Graduate School of The University of Texas.

Permanent address: 7506 Crossdraw  
Austin, Texas 78731

This thesis was typed by Jan Sheppard and Susie Thorn.